OPTIMAL TRANSPORTATION POLICY FOR PRODUCTION-DISTRIBUTION SYSTEM WITH MULTIPLE RETAILERS

by

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DEPARTMENT OF INDUSTRIAL AND MANAGEMENT ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
FEBRUARY, 1998

OPTIMAL TRANSPORTATION POLICY FOR PRODUCTION-DISTRIBUTION SYSTEM WITH MULTIPLE RETAILERS

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

by

Shishir Narain
(Roll No 9611417)

to

DEPARTMENT OF INDUSTRIAL AND MANAGEMENT ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

February, 1998

I.I.T. Kanpur
Submitted on 12.2.98

CERTIFICATE

This is to certify that the present work entitled "**OPTIMAL TRANSPORTATION POLICY FOR PRODUCTION-DISTRIBUTION SYSTEM WITH MULTIPLE RETAILERS**" has been carried out by Mr Shishir Narain (Roll No 9611417) under my supervision and that it has not been submitted elsewhere for a degree

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ACKNOWLEDGMENT

I take this opportunity to express my deepest gratitude to Dr A K Mittal for his able guidance an constant encouragement This I express not because it is customary to do so but from th experience I had from being associated with him

I am also thankful to the IME family that provided us a memorable stay here

This acknowledgment will seem incomplete if I don't appreciate the co-operation I got fro Neeraj, KD, Puneet, Rao, Sanjay and Suman

Thank You All

Shishir Nara

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ABSTRACT

The need to minimise the production, inventory and transportation costs has been long recognised by operations researchers Traditionally, the stages in logistical functions production at production site, transportation from production site to retailer site and consumption at retailer site have been optimised separately In the present research an integrated approach towards the above functions has been made

We consider production distribution system with a central production centre and multiple retailers Either one single product or multiple product-one for each retailer is being supplied through capacitated vehicles The demand is assumed to be continuous and deterministic The objective is to determine the optimal replenishment policy so as to maximise the overall profit by exploiting the cost trade-offs between individual logistical components

Two replenishment strategies single replenishment (i e one replenishment for a retailer in a single production cycle) and multiple replenishment have been analysed The focus of the present work is on analytical approach to solving production distribution problems The solution methodology follows the same line in both the cases It is first decomposed into retailerwise subproblem and then solved independently Finally these subproblems are integrated to form the simplified objective function The objective function is analysed for $3k$ ranges of the domain of n (number of production cycle) where k is the number of retailers In each range the local optimum can be obtained in polynomial time Then the best solution is obtained The computational complexity of both the algorithms are $O(k^2)$ subject to the condition that square root of a number is obtained in constant time Results of a computational study on randomly selected problems are also presented

Chapter 1

INTRODUCTION, LITERATURE SURVEY AND PROBLEM SETTING

1.1 Introduction

With the trend towards greater synergy between suppliers and industrial customers, most manufacturing enterprises are organised as networks of manufacturing and distribution sites that process the raw material and distribute the finished goods to customers Management of such networks (also referred to as "supply chains") has recently emerged as a major topic in Operations Research (Lee and Billington, 1993)

Optimal management of such networks requires careful analysis of the combined logistics problem The integrated model is used to study cost trade-offs between individual logistical components viz production, transportation and retail entity, as well as interrelations between various logistical functions This study can be utilised

i) to create an accurate representation of trade-offs encountered and

ii) to solve the combined problem

This type of combined logistic problem will be very useful for organizations, that own both the production and retail outlet facilities and thus are responsible for the replenishments and inventory management at both the sites The present research concerns the management of one segment of a supply chain by such an integrated approach decisions for both inventory and transportation functions are made simultaneously It will help in determining the production pattern and the shipping strategy for each retailer for maximum profit

1.2 Literature Survey

Although there is substantial literature on inventory and transportation management respectively, much less is available on the combined problem Initial study in this area began in mid-1980s

Federgruen and Zipkin (1984) integrated the allocation and routing problems within the same model by treating a single period, single item problem with random demands at the retailers No ordering cost is considered Their method is to first decompose the problem into two portions inventory allocation and vehicle routing The solution then is obtained by constructive and/or interchange heuristics

Burns et al (1985) developed a single item model with the objective of minimising transportation and inventory cost per unit time Production related costs are not included The transportation cost consists of a fixed charge per unit movement It is assumed that production is not synchronised with delivery and the cost of inventory accumulating before delivery is estimated accordingly The optimal delivery quantity is obtained by an economic order quantity type of analysis

The cost trade-offs between various logistical components on freight networks have been analyzed by Daganzo (1985) Networks involving direct shipping, shipping via consolidation terminal and a combination of terminal and direct shipping are considered by him He developed a simple optimizing method that simultaneously determines optimal routes and shipment sizes for networks He also identified conditions that indicate when networks involving direct shipments between many origins and destinations can be considered on a link by link basis

Benjamin (1989) investigated a single item version of the economic lot and delivery scheduling problem in which multiple deliveries may be made out of one batch He excluded from his objective function the cost of inventory accumulation before delivery and the additional inventories that would be required to avoid shortages when production interval (or batch) is not an integer multiple of the delivery interval (batch) These two implicit assumptions lead Benjamin to the conclusion that the problem may be solved optimally by independent EOQ type formulas for the production and delivery batches Unfortunately, the independent solutions generally will not have the integer multiple property that is assumed in the formulation

T W Chein (1991) dealt with profit oriented organisations that own both production and retail entities and face demand fluctuations He considered the case of stochastic demands and follows a probability distribution His research extends the classic Newsboy model by combining the production and logistical decisions He treated the production rate and shipment size as decision variables and calculated expected profits Monte Carlo experiments and sensitivity analysis are also performed to validate the model and view further insights into the effects of changes in parameter values on expected profit function

Anily and Federgruen (1990) consider a single item, deterministic model where the objective is to minimise average transportation and inventory costs over an infinite horizon They use a policy where a retailer can be assigned to different routes simultaneously, each route satisfying possibly a fraction of the total demand A partition method is used to derive lower bounds on the total cost It is then shown that these bounds become asymptotically tight when the number of demand points tend to infinity Anily and Federgruen (1993) extend this work to a two echelon distribution problem, where the central warehouse keeps system stocks instead of being a mere transhipment point Gallego and Simchi-Levi (1990) present a lower bound on the long run average cost over all inventory routing strategies for such systems They stress the benefits of direct shipping, whereas Hall (1992) argues that multiple-stop routing strategies can still provide large savings

For the case of deterministic demands for several products at several retailers, Vishwanathan and Mathur (1994) considered both inventory decisions and vehicle routes from a warehouse They obtain a stationary nested joint replenishment policy for the situation when replenishment intervals are power of two multiples of a planning period

V Anand (1997) has considered the combined logistics problem for a production distribution system with single production centre, single retailer outlet and a unique product under deterministic conditions He has formulated the problem and has derived closed form solution under uniform as well as non-uniform shipment policy

1.3 Problem Setting

In the present research, we consider the profit oriented organizations, which own both the production and retail entities The goal of the present research is to determine the optimal shipping strategies which maximises the overall profit by analysing the trade-offs that exist between production, inventory and transportation costs To obtain shipment policy, we formulate integrated model that consists of all the logistical components

The problem consists of producing a unique product (or multiple product, one for each retailer) at production site and shipping it directly to the various retailer using capacitated vehicles The demand at the outlet is assumed to be continuos and deterministic In case of multiple product, it is assumed that there is one identified retailer for that product

At the production site, the items are produced at a constant production rate The shipment to a retailer site is of equal volume However the volume and number of shipment will differ from retailer to retailer The costs incurred at the production site are cost of production set up and the cost of inventory holding

The capacity of transportation vehicle is limited There exists a fixed transportation cost for a shipment and a variable cost proportional to the number of units transported These charges are retailer dependent An inventory holding cost for the duration of transportation is also incurred

A regular revenue is obtained per unit of demand satisfied at the retailer outlet The demand not satisfied is considered lost and there exists a per unit shortage cost The inventory holding cost at the retail outlet is also taken in account

1.4 Organisation of thesis

In chapter 2, the case when only one replenishment is made to retailer site in a single production run is considered In chapter 3, the problem, in which more than one replenishment can be made to a retailer site in a production run is analysed In both the chapters the integrated problem is formulated, the constraints are identified and then a methodology is developed to solve the problem In chapter 4, we briefly discuss the conclusions and suggestions for future work

Chapter 2

PRODUCTION DISTRIBUTION SYSTEM WITH SINGLE REPLENISHMENT

In this chapter, we consider production distribution system with one production centre and multiple retailers The shipment from the production centre to the retail centre is made through vehicles (ships) with limited carrying capacity We further assume that only one shipment is made in each production cycle We consider the problem of finding the optimal shipment policy i e number of shipments per unit period, and the volume of shipment for each retailer, so as to maximise the overall profit after taking inventory, production and transportation costs into considerations

In section 2 1 we determine the general relationship between production pattern and shipment policy

2.1 Production Pattern and Shipment Policy

The system consists of a production site, a number of retailers and one capacitated vehicle for each retailer

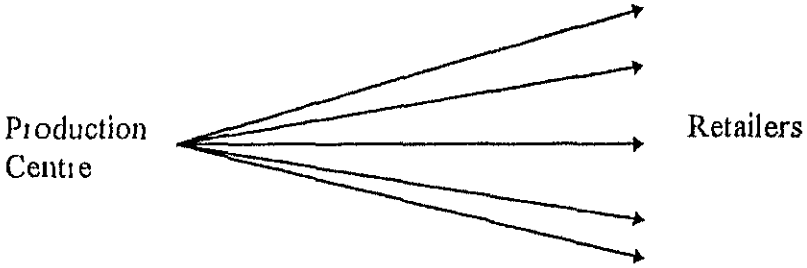


Figure 2.1 A Simple Production Distribution System

2.1.1 Assumptions

We consider this case with the following assumptions

1 Demand at a retailer is continuos and uniform

2 Production rate is greater than the cumulative demand rate of all the retailers

3 Demand not satisfied is lost

4 Net revenue is always positive

5 Only one shipment is made to a retailer in one set-up

6 One vehicle is available for shipment to each retailer, i e no scheduling of ships is involved [In section 2 8 we examine the case where the number of vehicles is not a constraint]

7 System is perpetual

We have considered the following cost and revenue for the purpose of optimisation as they are likely to be affected by the shipment decisions

A) Production site
  i) Production set up cost
  ii) Inventory holding cost
B) Transportation
  i) Cost of transportation
  ii) Cost of transportation inventory
C) Retail outlet
  i) Revenue received
  ii) Cost of holding inventory
  iii) Cost of lost demand

2 1 2 Notation

The following notation is used

Problem specification data

X_j	Demand rate per unit time period of retailer j
P	Production capacity of the production centre per unit time period
W_j	Maximum shipment capacity of the vehicle for retailer j
R	per unit net revenue (gross revenue- cost excluding inventory and transportation cost)
T_j	Transportation time for retailer j

C	Fixed cost per production set up
F_j, f_j	Fixed and per unit variable transportation cost of shipping to retailer j
s	per unit shortage cost
hp, ht	per unit holding cost per unit time period at production centre and during transportation respectively
hr_j	per unit holding cost per unit time period at retail outlet j
k	total number of retailers

__Decision variables__

V_j	Shipment size for retailer j
V_j^{opt}	Optimal shipment size for retailer j
n	number of set-ups in the given period
n_{opt}	Optimal value of n for the given range
θ_j	binary variable , 0 if replenishment is not made to retailer j 1 if replenishment is made to retailer j

2.1.3 Observations

When the system is perpetual, the following observation is made

Lemma (2.1) In a perpetual system, all cycles will be of equal length and total profit

Proof Consider one such system in which cycles are not of equal profit for a finite period of time 'L' Let it consist of n distinct cycles each with profit obtained per unit time is $C_1, C_2, \ldots C_n$ Let the length of the cycles be $T_1, T_2, \ldots T_n$

Total profit for the finite length of time L is $\sum_{i=1}^{n} C_i T_i$. (2.1)

Total profit per unit time period is $(\sum_{i=1}^{n} C_i T_i)/L$ (2.2)

There exists a maximum profit incurring cycle with profit C_{max} as max [C_i]

Then $\sum_{i=1}^{n} C_i T_i \le \sum_{i=1}^{n} C_{max} T_i \le C_{max} \sum_{i=1}^{n} T_i \le C_{max} L$. (2.3)

$$\Rightarrow \left(\sum_{i=1}^{n} C_i T_i\right)/L \le C_{max} \tag{2.4}$$

To maximise the total profit the system will have identical cycles of profit C_{max}

Hence the perpetual systems will have identical cycles

Let us consider the system for a unit time period in which demands at the retailer outlet j is X_j and the production capacity is P The shipment of size V_j is carried from the production site to the retail outlet j in each cycle, and there are n such cycles per unit period The total amount shipped to a retailer is less than or equal to the demand of the retailer in the unit period

With the above framework of the problem, the following observations are made

Lemma (2.2) When single shipment is made in a production cycle to each retailer then it is always advantageous to produce at the maximum rate for each retailer

Proof Let the production be done at the rate σ_j for the retailer j subject to the constraint that σ_j is lower than the production capacity P In that case we will have inventory holding cost at the production site

$$= hp\frac{1}{2}\frac{V_j^2}{\sigma_j}$$

The corresponding cost at full production rate P is $hp\dfrac{V_j^2}{2P}$

$$\text{As } \sigma_j < P \qquad hp\frac{1}{2}\frac{V_j^2}{\sigma_j} > hp\frac{V_j^2}{2P}$$

Hence the minimum inventory will be when $\sigma_j = P$

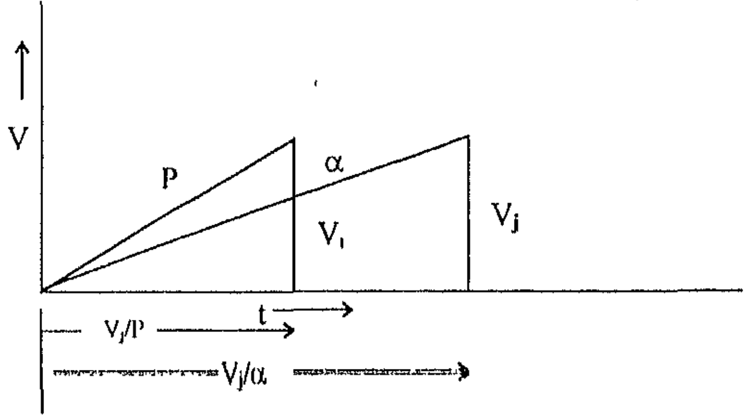


Figure 2.2 : Inventory Status Diagram at Production Centre for both Policies

Hence for the optimal shipment policy, the product will be manufactured at the maximum possible production rate

Lemma (2.3) In the optimal shipment policy, shipment to a retailer will be made as soon as the product has been produced (at the maximum rate of production)

Proof Let V_j be the optimal volume to the j^{th} retailer Let the shipment be made at an interval t subject to the condition that $V_j/P < t$ The inventory status diagram for this case is shown in fig 2 3

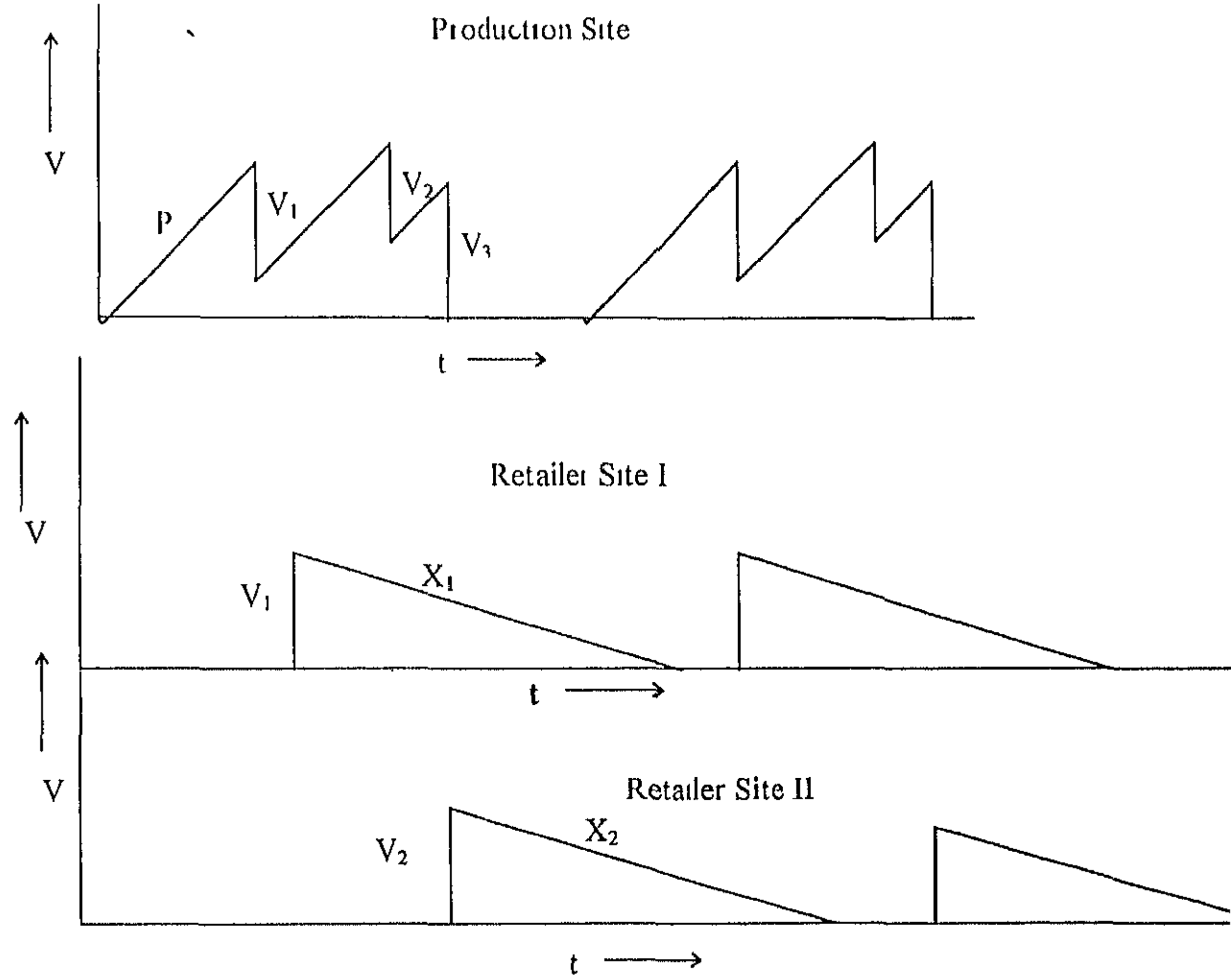


Figure 2.3 : Inventory Status Diagram for Independent Schedules

With the same retailer inventory status, same volumes and same number of cycles the corresponding inventory status diagram at the production site for the condition when shipment is made at made time t^* V_j/P is shown in fig 2.4.

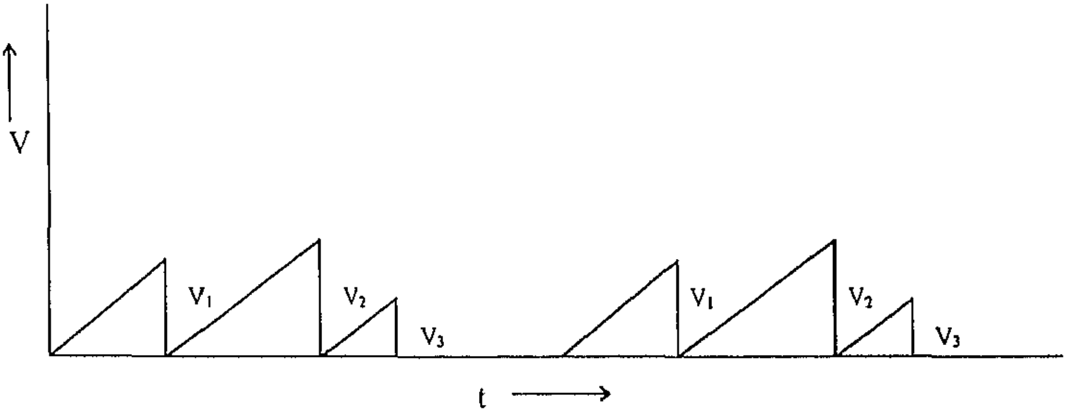


Figure 2.4 : Inventory Status Diagram for Synchronised Schedules

The area of the inventory status diagram gives the inventory holding cost. It is evident from the figures that the inventory holding cost is lesser in the second case as the area bounded between V and t is lesser than in first case, while all the other costs are same.

Hence the shipping policy that ships the required quantity as soon as it is produced is always better than delayed shipment.

In this section we develop the total profit function for such a policy.

2.2 Problem Formulation

In this section we formulate the net profit function per unit time period. The costs and revenues at production site, retail outlet and during transportation are calculated separately.

A) ***Production related cost***

The following figure represents the model at the production site. The relevant costs are cost of production set up and inventory costs.

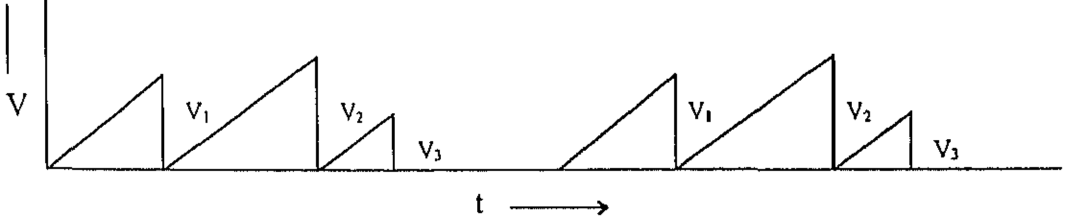


Figure 2.5 : Inventory Status Diagram at Production Site

1) Production set up cost There are n production set ups in a unit time. As C is the cost per production set up, production set up cost per unit time is nC (2.5)

2) Inventory holding cost The total inventory holding cost in one period can be computed by measuring the area of figure 2.5

The total inventory cost per unit period is

$$= \frac{n}{2} \frac{hp}{P} \sum_{j=1}^{k} V_j^2 \qquad (2.6)$$

B) Transportation related cost

1) Cost of transportation

A fixed freight charge F_j and a variable cost f_j per unit transported is incurred for the shipment to retailer j. Thus the total transportation cost is

$$n[F_j \sum_{j=1}^{k} \theta_j + \sum_{j=1}^{k} f_j V_j] \qquad (2.7)$$

where θ_j is a binary variable,

=1 when $V_j > 0$,

=0 when $V_j = 0$

2) Transportation inventory holding cost

If the transportation time is T_j for the j^{th} retailer then the total inventory holding cost during transportation over unit time period is

$$n\, ht \sum_{j=1}^{k} V_j T_j \qquad (2.8)$$

C) Retailer outlet

1) Revenue

The total amount of demand satisfied in one period is $n\sum_{j=1}^{k}V_j$ The total revenue generated is $n\,R\sum_{j=1}^{k}V_j$ (2.9)

2) Inventory cost The following figure represents the stock position at any time at the retailer outlet

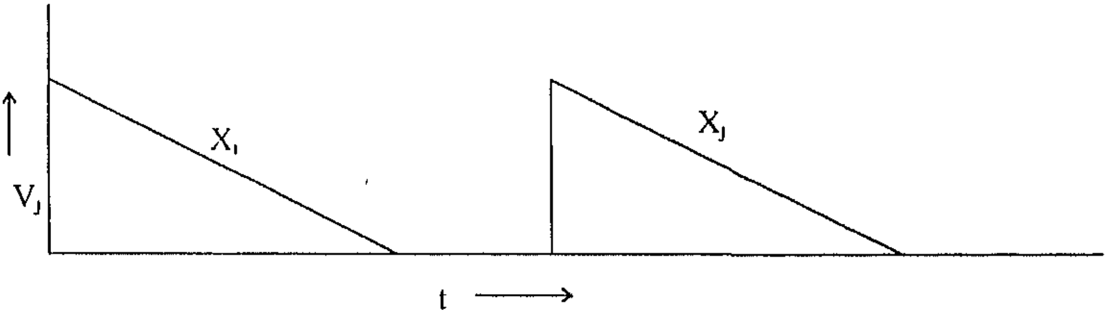


Figure 2.6 : Inventory Status Diagram at Retailer Site

If the per unit holding cost per unit time period at the retailer site j is hr_j then the total inventory holding cost per unit time period is

$$n\,\frac{1}{2}\sum_{j=1}^{k}\left(\frac{hr_jV_j^2}{X_j}\right) \qquad (2.10)$$

3) Shortage cost

The number of lost sales in one period at a retailer j is (X_j-nV_j) If the per unit shortage cost is s then the shortage cost per unit time period is

$$s\sum_{j=1}^{k}(X_j-nV_j) \qquad (2.11)$$

The profit function is obtained by subtracting all costs from the revenue obtained The profit is a function of n and V_j's Let us represent this function by Z Then

$$Z(n,V_j)=nR\sum_{j=1}^{k}V_j-nC-\frac{n}{2}\frac{hp}{P}\sum_{j=1}^{k}V_j^2-n[F_j\sum_{j=1}^{k}\theta_j+\sum_{j=1}^{k}f_jV_j]-n\,ht\sum_{j=1}^{k}V_jT_j$$

$$-n\,\frac{1}{2}\sum_{j=1}^{k}\left(\frac{hr_jV_j^2}{X_j}\right)-s\sum_{j=1}^{k}(X_j-nV_j) \qquad (2.12)$$

$$= n[(R+s)\sum_{j=1}^{k} V_j - \frac{hp}{2P}\sum_{j=1}^{k} V_j^2 - n\,\frac{1}{2}\sum_{j=1}^{k}\left(\frac{hr_j V_j^2}{X_j}\right) - ht\sum_{j=1}^{k} V_j T_j - \sum_{j=1}^{k} F_j\theta_j - \sum_{j=1}^{k} f_j V_j - C] -$$

$$s\sum_{j=1}^{k} X_j \qquad (2\ 13)$$

We will maximise Z to obtain the optimal policy The feasible policy has to further satisfy the following constraints,

1) Supply is less than or equal to demand

$$nV_j \leq X_j \qquad (2\ 14)$$

2) The volume shipped is less than or equal to the vehicle capacity

$$V_j \leq W_j \qquad (2\ 15)$$

3) Only one vehicle is allotted per retailer so that production cycle must be so spaced that the vehicle can return i e

$$n \leq \min[\frac{1}{\theta_j T_j}] \qquad (2\ 16)$$

4) non negativity constraint

$$V_j \geq 0,\ n \geq 0 \qquad (2\ 17)$$

2.3 Solution Methodology

Optimization of Z includes decision variables n and V_j We solve the problem by first getting optimal V_j^{opt} for each retailer as a function of n and then optimising it over the decision variables n to obtain optimal values i e n_{opt} and V_j^{op}

For a selected value of $n \neq 0$, the problem reduces to maximisation of $Z_1(V)$ We refer to this problem as **A**

Problem A

$$Z_1(V) = [(R+s)\sum_{j=1}^{k} V_j - \frac{hp}{2P}\sum_{j=1}^{k}\frac{V_j^2}{P} - \frac{1}{2}\sum_{j=1}^{k}\left(\frac{hr_j V_j^2}{X_j}\right) - ht\sum_{j=1}^{k} V_j T_j - \sum_{j=1}^{k} F_j\theta_j - \sum_{j=1}^{k} f_j V_j - C]$$

.. . (2 18)

subject to the constraints as given by Eqn 2 14, 2 15 and 2 17

It can be observed that objective function can be written as $Z(V)=\sum_{j=1}^{k} Z(V_j)$ Further there are no constraint binding the different decision variables together so the problem can be decomposed in solving independent subproblems each of the given type

Subproblem SP

The reduced j^{th} subproblem is

$$Z_2 = (R + s - f_j)V_j - \frac{hp}{2P} V_j^2 - \frac{hr_j}{2X_j} V_j^2 - ht\, V_j T_j - F\theta_j$$

$$= (R + s - f_j - ht\, T_j)\, V_j - \left(\frac{hp}{2P} + \frac{hr_j}{2X_j}\right) V_j^2 - F\theta_j \qquad (2.19)$$

Let $\alpha_j = \frac{hp}{2P} + \frac{hr_j}{2X_j}$ and $\beta_j = R + s - f_j - htT_j$ (2.20)

Then, substituting α_j and β_j we obtain

$$Z_2 = \beta_j V_j - \alpha_j V_j^2 - F\theta_j \qquad (2.21)$$

2.3.1 Observations

Case 1 If $\beta_j \le 0$, then $V_j = 0$, otherwise the losses increase on increasing replenishment size

Case 2 If $\beta_j > 0$ then $Z_2 = \begin{cases} 0 & V_j = 0 \\ \beta_j V_j - \alpha_j V_j^2 - F & V_j > 0 \end{cases}$ (2.22)

As we are now concerned with only one retailer we can drop the subscripts

From the equation 2.22 it is clear that only those values of V have to considered for which $Z_2 \ge 0$, because if for a $V > 0$, $Z_2 < 0$, then it implies that total cost (inventory holding cost and transportation cost) is higher than net revenue at that volume of shipment, and it will not be viable to make shipment at that values as $V=0$ is a better solution

Next we will calculate the range for V which $Z_2 \ge 0$ by finding the root of equation $Z_2 = 0$

For Z_2 to positive, V should take values between $\frac{\beta \pm \sqrt{\beta^2 - 4\alpha F}}{2\alpha}$ ... (2.23)

If $\beta^2 - 4\alpha F < 0$ then the roots are imaginary which implies that then Z_2 can not take positive value , so replenishment will not be made in that condition

2.3.2 Solution method for SP

One can look at the nature of the function which can be either of the two types as given below

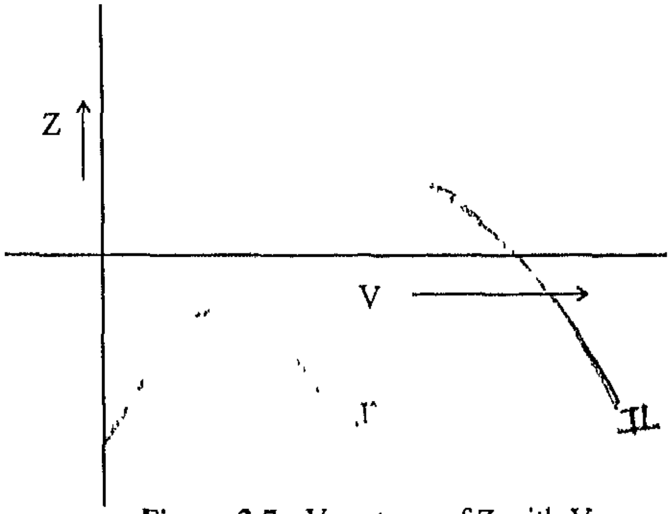


Figure 2.7 : Variation of Z with V.

From figure 2 7 it is clear that $V^{opt} = 0$ for case I Now we will consider case II

From unconstrained optimisation of Eqn 2 21 we get

$$\frac{dZ_2}{dV} = \beta - 2\alpha V \qquad (2.24)$$

$$\frac{d^2 Z_2}{dV^2} = -2\alpha$$ (< 0) so the function is concave and the V^* corresponds to the maxima of the function

So, the unconstrained optimal V is $V^* = \dfrac{\beta}{2\alpha}$ ...... (2.25)

Thus the value of permissible V varies with n as given below in the figure

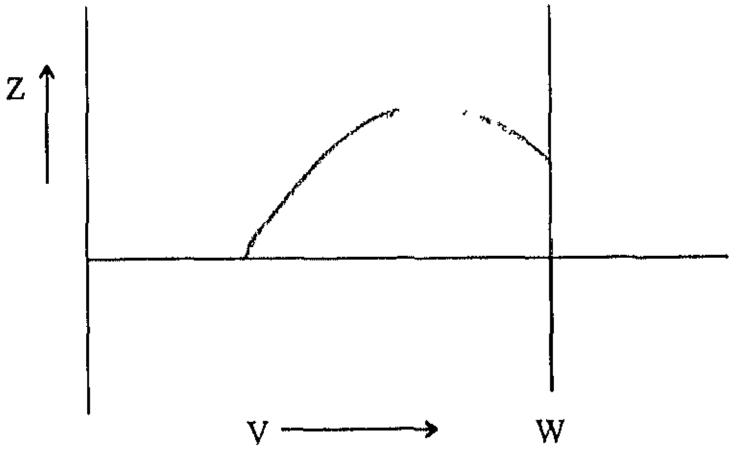


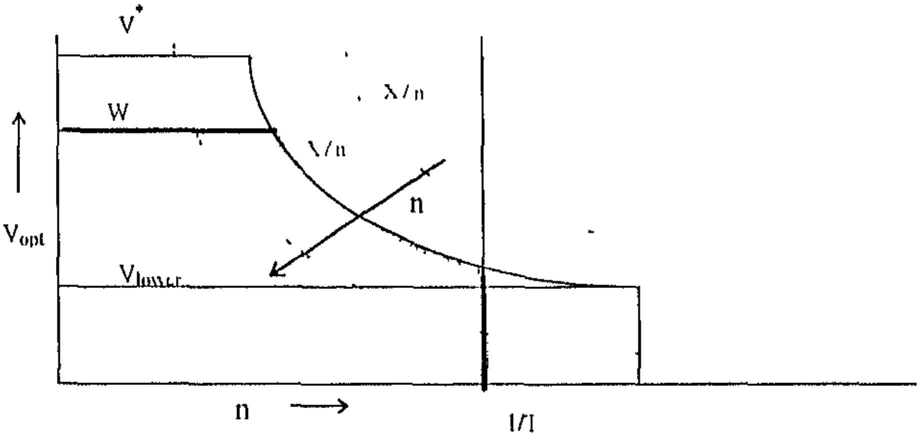
Figure 2.8 : Representation of Permissible Range of V for any Retailer

The value of V is bounded by W, the vehicle capacity on the upper side and by V_{lower} on the lower level

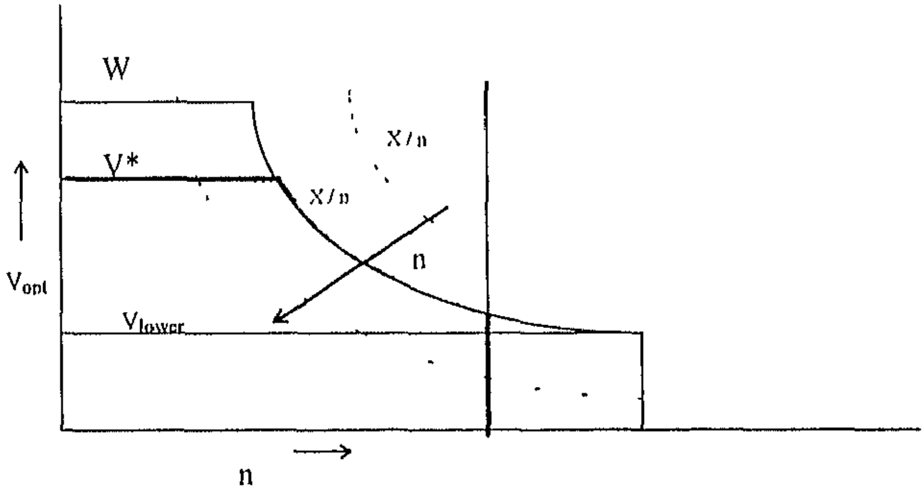
Let $\hat{V} = \min\left(V^*, \frac{X}{n}, W\right)$ then from the above analysis it is evident that

$$\begin{aligned} V_j^{opt} &= \hat{V} && \text{if } \hat{V} > V_{lower} \text{ and } n > 0 \\ &= 0 && \hat{V} < V_{lower} \text{ or } n = 0 \end{aligned} \qquad (2\ 26)$$

The figure representing V_j^{opt} for different values n is as given below



a). When V* > W



b) When V* < W

Figure 2.9 : Diagram Showing V_{opt} at any value of n

So, we have at most two breakpoints for each retailer with respect to n The first shows the point upto which the replenishment is of constant level (independent of n) min (V^*, W) The region between the first breakpoint and the second breakpoint gives those values of n for which full replenishment (X/n) is to be made Beyond the second breakpoint the size of optimal replenishment is zero

The first breakpoint, B_1, is achieved at $\min\left(\frac{X}{\min(V^*, W)}, \frac{1}{T}\right)$ (2 27)

The second breakpoint, B_2, is absent when $\frac{1}{T} < \frac{X}{\min(V^*, W)}$ otherwise it is attained at

$$\min\left(\frac{1}{T}, \frac{X}{V_{lower}}\right) \tag{2 28}$$

Here V_{lower} corresponds to the lower of the value obtained in Eqn 2 23

Thus at this stage we can compute V_j^{opt} of any retailer j for any value of n

2.3 3 Solution method for Problem A

The retailers for which optimal solution for all values of n is no shipment (i e V_j^{opt}=0) need not be considered any further Let G represent the set of index of all such retailers

Objective function :

For any given n, the objective function can be written as follows,

$$\sum_{j \in G_1} Z(V_j) + \sum_{j \in G_2} Z(V_j) \tag{2 29}$$

where

G_1= { all j such that V_j=min(V^*, W) }and

G_2= {all j such that V_j=X_j/n }

As at most there are two ranges for each retailer, the whole range of n (0 to ∞) can be partitioned into at most $2k$+1 distinct range For each, the objective function can be calculated by identifying the set G_1 and G_2 The objective function for a specific range will be one of the following type

A) Type I G_2={Φ}

When all retailers are being supplied at a level which is constant (independent of n)then the value of the function Z_1 is as follows,

$$Z_1 = n[\text{constant}] - s\sum_{j=1}^{k} X_j \tag{2 30}$$

Z is a linear function of n and hence within this range, n would take the upper limit of the range as the optimal value Thus for this range the local optimal value is the upper limit of the range

B) Type II : $G_1=\{\Phi\}$ *i e* all retailers are being supplied at full demand level (those having no replenishment have been already neglected)

Substituting $V_j = \frac{X_j}{n}$ in Eqn 2 13 we obtain

$$Z_4(n) \quad = R\sum_{j=1}^{k} X_j - \frac{hp}{2nP}\sum_{j=1}^{k} X_j^2 - \frac{1}{2n}\sum_{j=1}^{k} hr_j X_j - \sum_{j=1}^{k} f_j X_j - n\sum_{j=1}^{k} F_j - ht\sum_{j=1}^{k} X_j T_j - C$$

where $j \notin G$ (2 31)

Let $\theta_1 = R\sum_{j=1}^{k} X_j - \sum_{j=1}^{k} f_j X_j - ht\sum_{j=1}^{k} X_j T_j$, $\theta_2 = \frac{hp}{2P}\sum_{j=1}^{k} X_j^2 + \frac{1}{2}\sum_{j=1}^{k} hr_j X_j$ and $\theta_3 = \sum_{j=1}^{k} F_j + C$

where $j \notin G$

then the above equation can be expressed as

$$Z_4(n) = \theta_1 - (\theta_2/n) - n\theta_3 \tag{2 32}$$

where θ_2 and θ_3 are non-negative numbers

By differentiating Z_4 with respect to n, we obtain $n^* = \sqrt{\frac{\theta_2}{\theta_3}}$, (2 33)

and as the second derivative $(-\frac{\theta_2}{n^3})$ is always negative the given function is concave

Now we check the range for which full replenishment is being made

If n^* > upper limit of the range then n_{opt} = upper limit ,

if n^* < lower limit of the range then n_{opt} = lower limit else $n_{opt} = n^*$

C) Type III : G_1 and $G_2 \neq \phi$ *i e* Some retailers are being supplied at constant level (independent of n)and some are at full replenishment level

In this case the Eqn 2 14 can be written as

$$Z = n[Z_1(V_{j^*})] - s\sum_{j=1}^{k} X_j \qquad \text{where } j^* \in G_1, G_2 \tag{2 34}$$

The contribution due to set G_1 is a linear function with respect to n while the contribution due to set G_2 is of the nature as discussed in the above case (i e $Z_4(n) = \theta_1 - (\theta_2/n) - n\theta_3$)

Let the contribution from set G_1 be $n\theta_4 - \theta_5$ then we have

$$Z = n(\theta_1 - \theta_3) - (\theta_2/n) + \theta_1 - \theta_5$$

as in the above case

If $\theta_4-\theta_3 \geq 0$ then the function always increases with n, so in this case upper end of the range is chosen as the n_{opt}

If $\theta_4-\theta_3 < 0$ then on the same pattern as in type II, we obtain

$$n^* = \sqrt{\frac{\theta_2}{\theta_3 - \theta_4}} \tag{2 35}$$

Now we check the range for which full replenishment is being made

If n^* > upper limit of the range then n_{opt} = upper limit,

if n^* < lower limit of the range then n_{opt} = lower limit else $n_{opt} = n^*$

2.4 Algorithm

First we independently solve the retailerwise subproblem and then use this result to solve the total problem The detailed algorithm is given below

I Subproblem SP :

Initialise $j = 1$, G={Φ}

a) Calculate α_j and β_j where $\alpha_j = \frac{hp}{2P} + \frac{hr_j}{2X_j}$ and $\beta_j = R + s - f_j - hrT_j$

b) If $\beta_j < 0$ then assign j to set G and go to step g

c) Calculate $v\_star[j] = \frac{\beta_j}{2\alpha_j}$

and $v\_lower[j] = \frac{\beta_j - \sqrt{\beta_j^2 - 4\alpha_j F_j}}{2\alpha_j}$

if $\beta_j^2 < 4\alpha_j F_j$ then assign j to set G and go to step g

d) If $v\_star[j]$ W_j then $v\_star[j]$ W_j,

e) If $v\_star[j]$ $v\_lower[j]$ then assign j in set G and go to step g

f) Calculate the breakpoints as follows

$$B1[j] = \min\left[\frac{X_j}{v\_star[j]}, \frac{1}{T_j}\right]$$

if $B1[j] = \frac{1}{T_j}$ then $B2[j] = B1[j]$ else $B2[j] = \min\left[\frac{1}{T_j}, \frac{X_j}{v\_lower[j]}\right]$

g) Repeat the procedure for the next retailer

II Problem P

a) Combine arrays For $j=1$ to k and $j \notin G$ combine B1[j] and B2[j] and 0 and then arrange them in an ascending manner

b) Make pairs out of the above arranged list from each two consecutive numbers to construct ranges

c) For each range, check its type

Calculate G_1 and G_2

where G1= { all j such that $V_j=\min(V^*,W)$ }and

G_2= {all j such that $V_j=X_j/n$ }

If $G_1=\{\Phi\}$ then range type is Type I,

If $G_2=\{\Phi\}$ then range type is Type II else range type is Type III

d) Calculate n_{opt} for that range according to the range type

i) Type I $n_{opt} = range\_upper$

ii) Type II $n_{opt} = \sqrt{\dfrac{\dfrac{hp}{2P}\sum_{j=1}^{k} X_j^2 + \dfrac{1}{2}\sum_{j=1}^{k} h_j X_j}{\sum_{j=1}^{k} F_j + C}}$ where j $\notin$G

if $n_{opt} > range\_upper$ then $n_{opt} = range\_upper$,

if $n_{opt} < range\_lower$ then $n_{opt} = range\_lower$

iii)Type III Calculate the contribution to objective function due to set G_1

If $n\theta_4-\theta_3$ is the contribution, then if $\theta_4 > \sum_{j=1}^{k} F_j + C$, where j $\notin$G then

$n_{opt} = range\_upper$ else

Calculate $\theta_2=\dfrac{hp}{2P}\sum_{j=1}^{k} X_j^2 + \dfrac{1}{2}\sum_{j=1}^{k} h_j X_j$ where $j \in G_2$ then

$n_{opt}=\sqrt{\dfrac{\theta_2}{\sum_{j=1}^{k} F_j + C - \theta_4}}$ where $j \notin$G

if n_{opt} > range_upper then n_{opt} = range_upper,

if n_{opt} < range_lower then n_{opt} = range_lower

e) At each of these n_{opt} the corresponding V_j's are calculated if $n_{opt} < B1[j]$ then $V_j^{opt} = \min(V^*, W_j)$

if n_{opt} lies between $B1[j]$ and $B2[j]$ then $V_j^{opt} = \frac{X_j}{n_{opt}}$

if $n_{opt} > B2[j]$ then $V_j^{opt} = 0$

f) At each n_{opt} calculate the objective function value and select n_{opt}^* as n_{opt} for which objective function value is maximum

2.5 Flow Chart

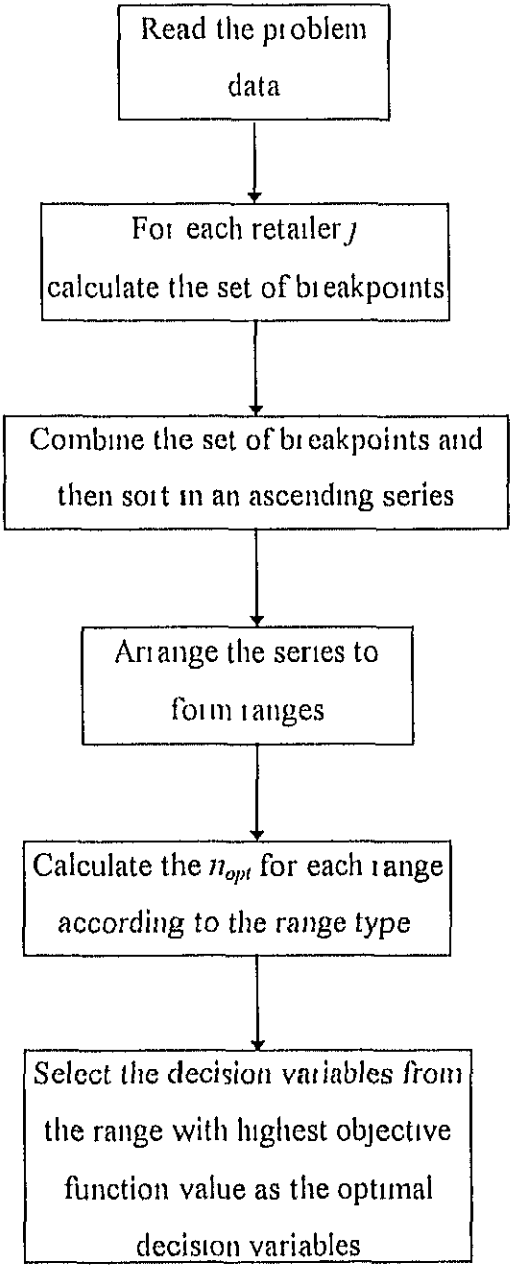

Read the problem data
For each retailer j calculate the set of breakpoints
Combine the set of breakpoints and then sort in an ascending series
Arrange the series to form ranges
Calculate the n_{opt} for each range according to the range type
Select the decision variables from the range with highest objective function value as the optimal decision variables

Figure 2.10 : Flow Chart for the Algorithm

2.6 Complexity Analysis

Let k indicate the number of retailers In the first part, **SP**, each of the five step requires a constant number of operations (addition, subtraction, comparison, division etc) so the total number of operations for one retailer is a constant, say L This procedure is repeated k number of times thus the number of operations required is Lk, which also assumes that square root operation takes a constant amount of time i e operation time is independent of problem size

For the problem **P** we have to first sort the $2k$ ranges which requires ($2k \log 2k$) steps Now for each range we make k evaluations to find the optimal decision variables This step requires $2k^2$ steps

Thus the problem requires $Lk+2k \log 2k + 2k^2$ steps Hence the complexity of the above algorithm is $O(k^2)$ i e it is a polynomial function of number of retailers

2.7 Example

To illustrate the above analysis we consider a numerical example with the following data

R= \$615 per unit, c= \$600 per unit, P= 5000 units per year, s= \$10 per unit, hp= \$80 per unit per year, hr = \$50 per unit per year (for all retailers), ht = \$15 per unit per year, k= 4, $F_1= F_2= F_3= F_4=$ \$1000 per transportation , $f_1= f_2= f_3= f_4=$ \$1 per unit

$X_1=$ 600 units per year, $X_2=$ 900 units per year, $X_3=$ 1200 units per year, $X_4=$ 1000 units per year, T_1=0 05 year, T_2=0 08 year, T_3=0 125 year, T_4=0 10 year $W_1= W_2= W_3= W_4$=450 units

Solution : As explained in the above section we can get a table like the given below

Retailer no →	1	2	3	4
V*	234	318 6	383 7	340 91
V_{lower}	47 91	47 38	48,22	47 79
min (V*,W)	234	318 6	383 7	340 19
B_1	2 56	2 82	3 13	2 93
B_2	12 5	12 5	8	10

Table 2.1: Subproblem Calculation

On arranging the combined set of breakpoints in an ascending series and then making pair out of the adjacent numbers we get seven ranges The subsequent analysis is presented in the table below

Range	Type	n_{opt}	Z
0 - 2 56	I	2 55	-10291 65
2 56 - 2 82	III $G_1=\{2,3,4\}$ $G_2=\{1\}$	2 81	-7742
2 82 - 2 93	III $G_1=\{3,4\}$, $G_2=\{1,2\}$	2 92	-6665
2 93 - 3 13	III $G_1=\{3\}$, $G_2=\{1,2,4\}$	3 13	-4863
3 13 - 8 0	II $G_2=\{1,2,3,4\}$	5 44	2447
8 0 - 10 0	II $G_2=\{1,2,4\}$	8 0	-14012
10 - 12 5	II $G_2=\{1,2\}$	10 0	-17994

Table 2.2 : Problem Calculation

From the above table it is evident that the optimal value of n is 5 44 and the replenishment sizes for the retailers are 110 27, 165 41, 220 54 and 183 79 units respectively

2.8 Problem with Unlimited Number of Vehicles

The same problem that has been considered in the above sections can be solved for the case when the number of vehicles is not restricted This implies that the constraint that

$n \leq \min[\frac{1}{\theta T_i}]$ need not be satisfied

Due to this relaxation, the breakpoint calculation is altered slightly

The first breakpoint, B_1, is achieved at $\left(\frac{X}{\min(V^*, W)}\right)$. (2 36)

The second breakpoint, B_2, is attained at $\min\left(\frac{X}{V_{lower}}\right)$. (2 37)

Here V_{lower} corresponds to the lower of the value obtained in Eqn 2.23

The rest of the solution methodology is exactly the same

Chapter 3

PRODUCTION DISTRIBUTION SYSTEM WITH MULTIPLE REPLENISHMENTS

In this chapter, we consider the production distribution system where multiple shipments to a retailers in a single production cycle are permitted We consider two kinds of production policies In the first situation, only one production setup for each retailer, in each cycle is permitted and the production takes place at the maximum production rate P Such a situation may occur when the production is in identified batches for each retailer (i e order processing) or multiple products are being produced, one for each retailer In the second situation we consider that simultaneous production take place for each retailer and shipments are made to each of the retailers throughout the cycle Such a case may refer to a situation where a common product is being produced to be shipped to all retailers or to the case when all the products are being produced simultaneously at the rate consistent with their total replenishment in the cycle

As in the previous chapter, we first determine the general behavior of production and demand patterns for the above situations and then formulate the profit function by considering all relevant costs and revenues Next, we optimise the profit function subject to certain constraint to obtain the optimal shipping policy

Assumptions

The same set of assumptions as in section 2 1 have been considered except the assumption of only one shipment per production setup has been relaxed

Notation

In addition to the notation described in section 2 1 we will use the following notation

m_j number of replenishment in a single setup for retailer j

$$\phi_j \quad \begin{cases} 0 & if \quad hp \geq hr_j \\ 1 & if \quad hp < hr_j \end{cases}$$

The nature of the shipment policy will depend upon the relative cost of maintaining inventory at the production site (hp) and at the retailer site (hr) We consider the following two cases

Case I If the cost of maintaining inventory at production site is larger than the retailer site (i e $hp > hr_j$) then optimal number of shipments of fixed size will be made to the retailer centre as soon as possible so as to reduce the inventory at production site at the cost of increasing it at the retailer site

Case II If the cost of maintaining inventory at retailer site is greater than that at the production site (i e $hp \quad hr_j$) then the shipments will be delayed as far as possible so as to reduce the inventory cost at the retailer site at the expense of increasing inventory holding cost at production site

Next we consider the production policy in which only one production setup for each retailer is permitted in a single cycle

3.1. Single Production Setup Multiple Replenishment for each Retailer (SPS)

In this case we consider the situation when total order of a retailer is produced at the maximum rate of production only once in a cycle The product is shipped in shipments of equal size that are spaced at equal intervals throughout the cycle We shall formulate the net profit function for this case The costs and revenues at production site, retail outlet and during transportation are calculated separately

3.1.1 Problem Formulation

In this section we formulate the net profit function per unit time period The costs and revenues at production site, retail outlet and during transportation are calculated separately

A) ***Production related cost***

The following figures represent the models at the production site for case I and case II respectively The relevant costs are cost of production setup and inventory costs

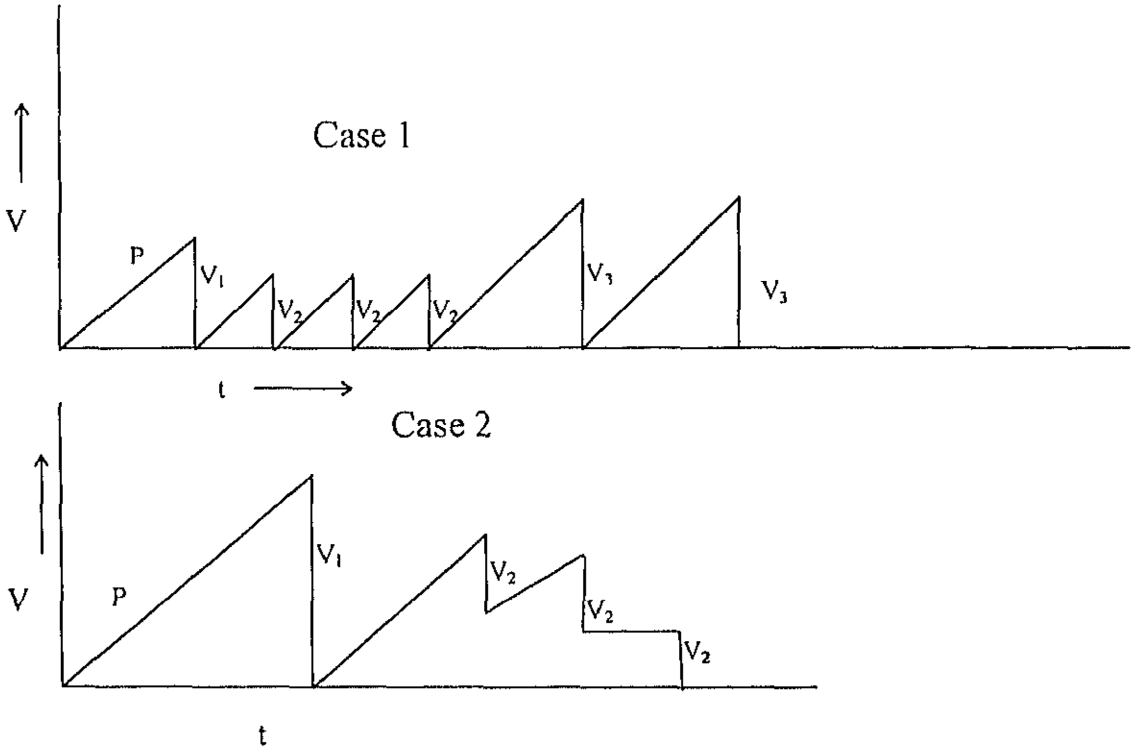


Figure 3.1: Inventory Status Diagram at Production site

1) Production set up cost There are n production setups in a unit time As the cost of set up is C, the total production set up cost per unit time is nC (3 1)

2) Inventory holding cost The total inventory holding cost in one period can be computed by calculating the area of the above figure If the inventory holding cost is hp then the total inventory cost per unit period for case I is,

$$n\,hp\,\frac{1}{2}\sum_{j=1}^{k}\frac{m_j V_j^2}{P}$$

$$=\frac{n}{2}\,\frac{hp}{P}\sum_{j=1}^{k} m_j V_j^2 \qquad (3\ 2a)$$

and for case II is

$$\frac{nhp}{2}\sum_{j=1}^{k} m_j \frac{V_j^2}{X_j}\left[(m_j+1)-\left(\frac{m_j X_j}{P}\right)\right] \text{ [See Appendix-A 1]} \qquad (3\ 2b)$$

B) Transportation related cost

1) Cost of transportation

A fixed freight charge F_j and a variable cost f_j per unit transported is incurred depending upon the retailer site Thus the total transportation cost is

$$n[\sum_{j=1}^{k} F_j m_j + \sum_{j=1}^{k} m_j f_j V_j] \tag{3 3}$$

2) Transportation inventory holding cost

If the transportation time is T_j for the j^{th} retailer and the transportation inventory holding cost is ht then the total inventory holding cost during transportation over the unit time period is

$$n\, ht \sum_{j=1}^{k} m_j V_j T_j \tag{3 4}$$

C) Revenue and cost at the retailer site

1) Revenue

The total amount of demand satisfied in one period is $n\sum_{j=1}^{k} m_j V_j$ The total revenue generated is

$$n\, R \sum_{j=1}^{k} m_j V_j \tag{3 5}$$

2) Inventory cost

The following figures represents the stock position at any time at the retailer outlet for case I and case II respectively

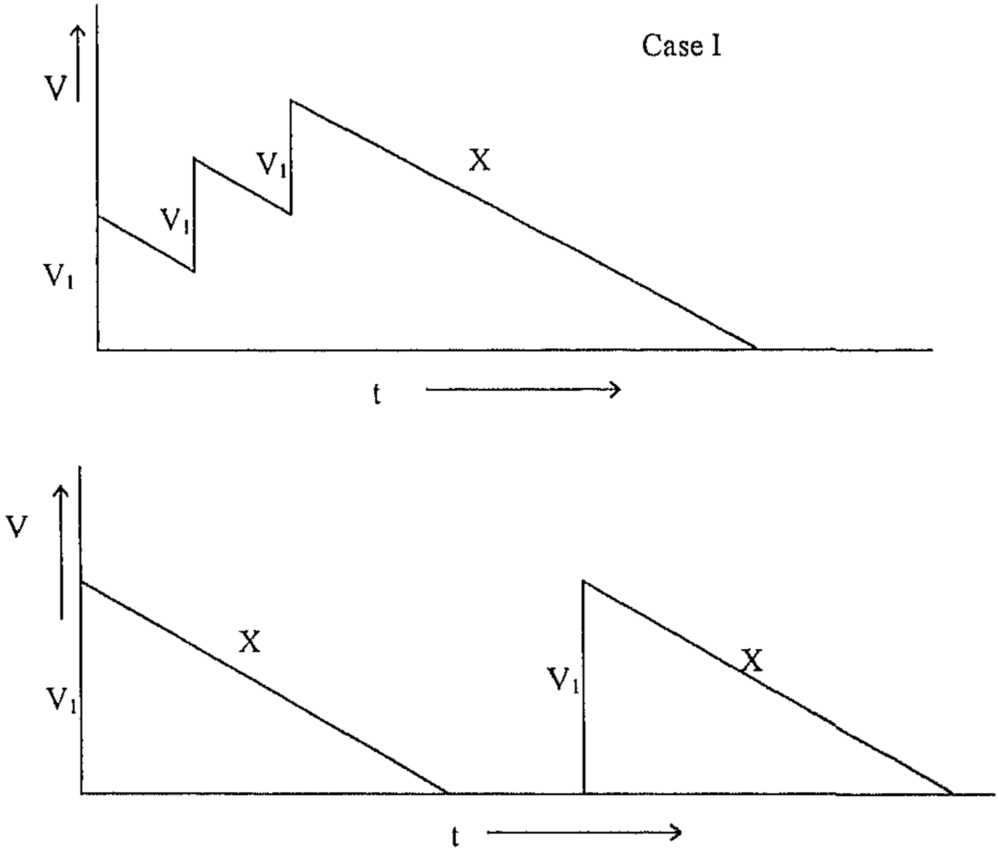


Figure 3.2 : *Inventory Status Diagram at Retailer Site*

If the per unit holding cost at the retailer's site is hr_j, the total inventory holding cost per unit time period is

for case I

$$n\,\frac{1}{2}\sum_{j=1}^{k} hr_j m_j V_j^2\left(\frac{m_j}{X_j}-\frac{m_j-1}{P}\right) \qquad \text{[See Appendix-A 2]}$$

and for case II

$$n\,\frac{1}{2}\sum_{j=1}^{k}\frac{hr_j m_j V_j^2}{X_j} \qquad (3.6)$$

3) Shortage cost. The number of lost sales in one period at a retailer j is $(X_j - n\, m_j V_j)$. If the per unit shortage cost is s then the shortage cost per unit time period is

$$s\sum_{j=1}^{k}\left(X_j - n m_j V_j\right) \qquad (3.7)$$

The profit function is obtained by subtracting all costs from the revenue obtained. The profit is a function of n, m_j and V_j's. Let us represent this function by Z_1. Then we describe this problem, denoting it as **P**, in the next section.

3.2 Problem P

$$Z_I(n,\ m_j\ V_j) = n\,R \sum_{j=1}^{k} m_j V_j - n\,C - \frac{n}{2}\ \frac{hp}{P} \sum_{j=1}^{k}\left(1-\phi_j\right) m_j V_j^2 - n$$

$$\frac{1}{2}\sum_{j=1}^{k}\left(1-\phi_j\right) hr_j m_j V_j^2\left(\frac{m_j}{X_j}-\frac{m_j-1}{P}\right) - \frac{nhp}{2}\sum_{j=1}^{k}\phi_j m_j \frac{V_j^2}{X_j}\left[\left(m_j+1\right)-\left(\frac{m_j X_j}{P}\right)\right] -$$

$$n\frac{1}{2}\sum_{j=1}^{k}\frac{\phi_j hr_j m_j V_j^2}{X_j} - n[\sum_{j=1}^{k} F_j m_j + \sum_{j=1}^{k} m_j f_j V_j] - n\,ht \sum_{j=1}^{k} m_j V_j T_j - s\sum_{j=1}^{k}\left(X_j - nm_j V_j\right)$$

(3 8)

We will maximise Z_I to obtain the optimal policy The feasible policy has to further satisfy the following constraints

1) Supply is less than or equal to demand

$$n\,m_j V_j \le X_j \qquad (3\ 9)$$

2) The volume shipped is less than or equal to the vehicle capacity

$$V_j \le W_j \qquad (3\ 10)$$

3) Only one vehicle is allotted per retailer so the production cycle must be so spaced that the vehicle can return to take the new replenishment

$$nm_j \le \frac{1}{T_j} \qquad (3\ 11)$$

4) non negativity constraint

$$n,\ m_j\ V_j \ge 0 \qquad (3\ 12)$$

The function $Z_I(n,\ m_j\ V_j)$ can be rewritten as follows

$$Z_I(n,\ m_j\ V_j) = n\ \{R \sum_{j=1}^{k} m_j V_j - C - \frac{hp}{2P}\sum_{j=1}^{k}\left(1-\phi_j\right) m_j V_j^2 -$$

$$\frac{1}{2}\sum_{j=1}^{k}\left(1-\phi_j\right) hr_j m_j V_j^2\left(\frac{m_j}{X_j}-\frac{m_j-1}{P}\right) - \frac{hp}{2}\sum_{j=1}^{k}\phi_j m_j \frac{V_j^2}{X_j}\left[\left(m_j+1\right)-\left(\frac{m_j X_j}{P}\right)\right] -$$

$$\frac{1}{2}\sum_{j=1}^{k}\frac{\phi_j hr_j m_j V_j^2}{X_j} - [\sum_{j=1}^{k} F_j m_j + \sum_{j=1}^{k} m_j f_j V_j] - ht \sum_{j=1}^{k} m_j V_j T_j + s\sum_{j=1}^{k} m_j V_j\} - s\sum_{j=1}^{k} X_j \qquad (3\ 13)$$

Thus $\quad Z_I(n,\ m_j\ V_j) = n\sum_{j=1}^{k} Z_{11}\left(m_j, V_j\right)$

$$Z_{11}(m_j,\ V_j) = (R+s)\sum_{j=1}^{k} m_j V_j - \frac{hp}{2P}\sum_{j=1}^{k}\left(1-\phi_j\right)m_j V_j^2 - \frac{1}{2}\sum_{j=1}^{k}\left(1-\phi_j\right) hr_j m_j V_j^2\left(\frac{m_j}{X_j}-\frac{m_j-1}{P}\right) -$$

$$\frac{hp}{2}\sum_{j=1}^{k}\phi_j m_j \frac{V_j^2}{X_j}\left[\left(m_j+1\right)-\left(\frac{m_j X_j}{P}\right)\right] - \frac{1}{2}\sum_{j=1}^{k}\frac{\phi_j hr_j m_j V_j^2}{X_j} - ht\sum_{j=1}^{k} m_j V_j T_j - \sum_{j=1}^{k} F_j m_j - \sum_{j=1}^{k} m_j f_j V_j$$

(3 14)

subject to

1) $m_j V_j \leq X_j/n$ (3 15)

2) $0 \leq V_j \leq W$ (3 16)

3) $0 \leq m_j \leq \dfrac{1}{nT_j}$ (3 17)

Further in this equation as no constraint binds different decision variables together so the problem can be decomposed into k independent problem, one for each retailer

3 2 1 Subproblem SP

$$Z_{12}(m_j,\ V_j) = (R+s-f_j-ht\ T_j)\ m_j V_j - \left(1-\phi_j\right)\left(\frac{hp}{2P}+\frac{hr}{2}\left(\frac{m_j}{X_j}-\frac{m-1}{P}\right)\right) m_j V_j^2 -$$

$$\phi_j\left(\frac{hp+hr_j}{2X_j}\right)\left[\frac{m_j\left(P-X_j\right)}{2PX_j}+1\right]\frac{m_j V_j^2}{X_j} - F_j m_j$$

$$= (R+s-f_j-htT_j)\ m_j V_j - \left(1-\phi_j\right) m_j V_j^2\left(\frac{hp}{2P}+\frac{hr}{2P}\right) - F_j m_j -$$

$$\left(1-\phi_j\right) m_j^2 V_j^2 hr\left(\frac{P-X_j}{2PX_j}\right) - \phi_j\left(\frac{hp+hr_j}{2X_j}\right)m_j V_j^2 - \phi_j m_j^2 V_j^2 hp\left(\frac{P-X_j}{2PX_j}\right) \qquad (3.18)$$

Let $\alpha_j = \left(1-\phi_j\right)\left(\frac{hp}{2P}+\frac{hr_j}{2P}\right)+\phi_j\left(\frac{hp+hr_j}{2X_j}\right)$, $\beta_j = R+s-f_j-ht\ T_j$, and

$\gamma_j = \left(1-\phi_j\right)\left(\frac{P-X_j}{2PX_j}\right)hr_j + \phi_j\left(\frac{P-X_j}{2PX_j}\right)hp$

It may be noted that α_j and $\gamma_j > 0$

On substituting α_j, β_j and γ_j we obtain the following expression

$$Z_{12}(m_j,\ V_j) = \beta_j m_j V_j - \alpha_j m_j V_j^2 - m_j^2 V_j^2 \gamma_j - F_j m_j \qquad (3\ 19)$$

3.3 Solution Methodology

As the objective function is separable in k subproblems for a fixed value of n, we will solve the problem as follows

i) we will study the nature of optimal solution for the subproblem **SP** for a fixed value of n

ii) we will optimise the objective function with respect to the decision variable n

3.3 1 Observations

Case A

If $\beta_j \leq 0$, then m_j and $V_j^{opt} = 0$, as for any $V_j > 0$ the profit will be less than zero

$$V_j^{opt} = 0 \quad \Rightarrow m = 0$$

Case B

If $\beta_j > 0$ then the equation 3 19 reduces to

$$Z_{12}(m_j, V_j) = 0 \qquad \text{for } V_j = 0$$

$$= \beta_j m_j V_j - \alpha_j m_j V_j^2 - m_j^2 V_j^2 \gamma_j - F_j m_j \qquad \text{for } V_j > 0 \qquad (3\ 20)$$

If $V_j = 0$ then $m_j^{opt} = 0$

Further if $V_j > 0$, then there will be atleast one shipment per cycle and hence $m_j \geq 1$

Optimisation for a Fixed Value of *n*

As we are now concerned with only one retailer we can drop the subscripts and will consider the function

$$Z_{12}(m, V) = \begin{cases} \beta m V - \alpha m V^2 - m^2 V^2 \gamma - mF & V > 0, m \geq 1 \\ 0 & V = 0, m = 0 \end{cases}$$

Next we shall examine this function in the domain $V > 0$, $m \geq 1$

First we will calculate all the points which satisfy the first order necessary condition for any point to be maximum Among the points so obtained, we will compute the set of local maximum points, if any The first order necessary condition for a local maxima (or mimina) are

$$\frac{\partial Z_{12}}{\partial V} = \beta m - 2\alpha m V - 2\gamma m^2 V = 0 \qquad (3\ 21)$$

$$\frac{\partial Z_{12}}{\partial m} = \beta V - \alpha V^2 - 2\gamma m V^2 - F = 0 \tag{3 22}$$

As we are considering only the region where $V > 0$ and $m > 0$ ($m \geq 1$) we obtain from equation 3 21 the point which may be a local maxima (or minima) iff

$$V^* = \frac{\beta}{2(\alpha + \gamma m)} \tag{3 23}$$

Further substituting it in eqn 3 22 we obtain

$$m^* = \left\{ \frac{-\sqrt{\frac{\beta^2 \alpha}{4F}} - \alpha}{\gamma}, \frac{+\sqrt{\frac{\beta^2 \alpha}{4F}} - \alpha}{\gamma} \right\} \tag{3 24}$$

As we interested only in the points where $V > 0$, $m > 0$, there is at most one point

$$(m^*, V^*) = \left(\frac{\sqrt{\frac{\beta^2 \alpha}{4F}} - \alpha}{\gamma}, \frac{\beta}{2(\alpha + \gamma m)} \right)$$

which has the potential for a maxima

Further substituting m^* in eqn 3 23 we obtain

$$(m^*, V^*) = \left(\frac{\sqrt{\frac{\beta^2 \alpha}{4F}} - \alpha}{\gamma}, \sqrt{\frac{F}{\alpha}} \right) \tag{3 25}$$

The m^* and V^* should additionally satisfy the constraints that $V^* < W$ and $m^* V^* < X/n$

So the region where m^* and V^* can lie is

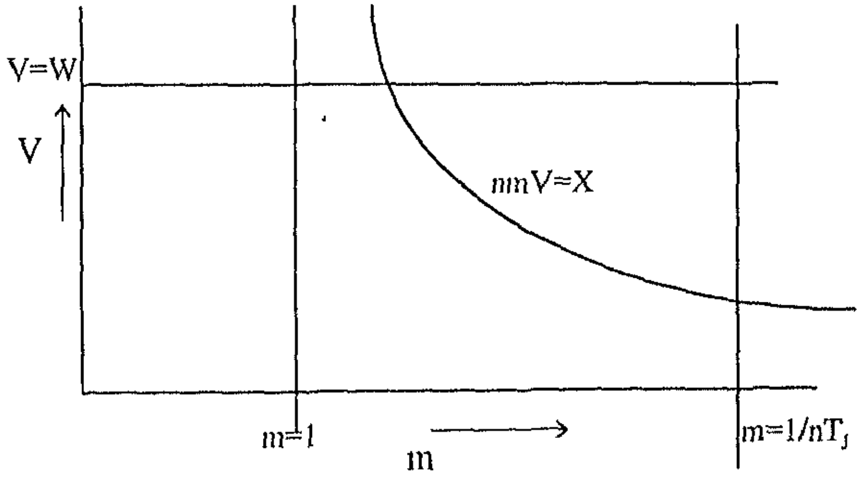


Figure 3.3 : Feasible region of m* and V*

Now if $Z_{12}(m^*, V^*) > 0$ and it also lies within the interior region, then the point (m^*, V^*) gives the maximum, because at m 1 and $V=0$ the value of $Z_{12} < 0$ and, there is only one point (m^*, V^*) which satisfies the necessary condition for a local maximum or minimum

Hence we conclude that the optimum solution will be one of the following type

a) $V = 0$, $m = 0$ if $\beta \leq 0$ or $Z_{12}(m^*, V^*) < 0$

b) (m^*, V^*) if $Z_{12}(m^*, V^*) > 0$ and (m^*, V^*) lies within the boundary

c) At the boundary (if it lies outside the boundary)

3 3.2 Search at the Boundary

1) At the boundary $m = 1$

From eqn 3 23 we get $V^* = \dfrac{\beta}{2(\alpha + \gamma)}$ (3 26)

but V^{opt} should also lie between W, the vehicle capacity, and the oversupply constraint i e $V^{opt} = \min\left(\dfrac{\beta}{2(\alpha + \gamma)}, W, X/n\right)$

2) At the boundary $V = W$, $m \geq 1$ and $mV \leq X/n$

Substituting $V = W$ in eqn 3 22 we obtain

$$m^* = \frac{\beta W - \alpha W^2 - F}{2\gamma W^2} \tag{3 27}$$

if $(\beta W - \alpha W^2 - F) < 0$ then $m^* = 1$ as $m \geq 1$

if $(\beta W - \alpha W^2 - F) \geq 0$ and if $m^* W > X/n$ then

$m^* = X/(nW)$

$$\text{so } m^{opt} = \begin{cases} 1 & \beta W - \alpha W^2 - F < 0 \\ \dfrac{\beta W - \alpha W^2 - F}{2W^2\gamma} & \beta W - \alpha W^2 - F \geq 0 \\ \dfrac{X}{nW} & \dfrac{\beta W - \alpha W^2 - F}{2\gamma W} \geq \dfrac{X}{n} \end{cases}$$

3) At the boundary $mV = X/n$

We set $mV = X/n$ in Eqn 3 21 to get

$$Z_{12} = \frac{\beta X}{n} - \frac{\alpha X^2}{n^2 m} - \frac{\gamma X^2}{n^2} - mF \qquad (3.28)$$

$$\frac{\partial Z_{12}}{\partial n} = \frac{\alpha X^2}{n^2 m^2} - F$$

$$\frac{\partial^2 Z_{12}}{\partial m^2} = -\frac{\alpha X^2}{n^2 m^3} \qquad (<0) \text{ Hence the function is concave} \qquad (3.29)$$

Equating the first derivative to zero we get

$$m^* = \frac{X}{n}\sqrt{\frac{\alpha}{F}} \qquad (3.30)$$

$$V^* = \sqrt{\frac{F}{\alpha}}$$

if $V^* > W$ then $V^* = W$ and $m^* = X/(nW)$

So $m^{opt} = \min(\frac{X}{n}\sqrt{\frac{\alpha}{F}}, X/(nW))$ and $V^{opt} = \min\left(\sqrt{\frac{F}{\alpha}}, W\right)$

4) At the boundary $m = \frac{1}{nT'_j}$

On substituting this value of m in eqn 3.23 we obtain

$$V^* = \min\left(XT'_j, \frac{n\beta T'_j}{2(n\alpha T'_j + \gamma)}\right)$$

Thus there are at most six points which can be the candidate for the maximum, four on the boundaries, one at origin and the last being the unconstrained optimal

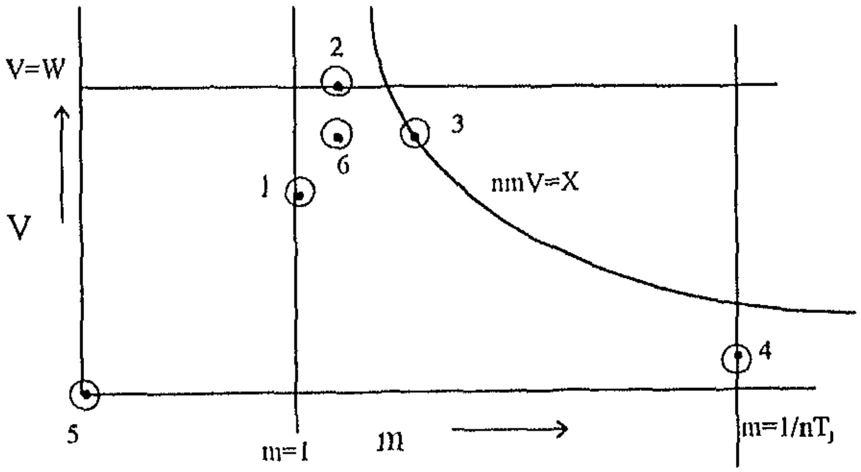


Figure 3.4 : Candidate Points for Maximum in the Feasible Region

At these points, the various distinct values which m^* and V^* can take is given below The number in bracket indicates the corresponding point on Fig 3 4

1 $\dfrac{\sqrt{\dfrac{\beta^2\alpha}{4F}}-\alpha}{\gamma}, \sqrt{\dfrac{F}{\alpha}}$ (6)

2 $\dfrac{X}{n}\sqrt{\dfrac{\alpha}{F}}, \sqrt{\dfrac{F}{\alpha}}$ (2)

3 $\dfrac{\beta W-\alpha W^2-F}{2\gamma W^2}, W$ (2)

3 $\dfrac{\beta W-\alpha W^2-F}{2\gamma W^2}, W$ (2)

4 $\dfrac{X}{nW}, W$ (2 or 3)

5 $1, W$ (1 or 2)

6 $1, \dfrac{\beta}{2(\alpha+\gamma)}$ (1)

7 $1, \dfrac{X}{n}$ (1)

8 $\dfrac{1}{T_j}, XT_j$ (4)

9 $\dfrac{1}{T_j}, \dfrac{\beta T_j}{2(\alpha T_j+\gamma)}$ (4)

Now if we relax the condition that there is only one vehicle, then the m^{opt} and V^{opt} will occur at one of the first seven points only

3.3.3 Optimisation with respect to *n*

For the sake of simplicity we will consider the problem when the condition of one vehicle for each retailer is relaxed i e the constraint given by Eqn 3 17 is not effective

To examine the effect of change of value of n on the optimal solution, we consider the following three cases

Case I $m^* \geq 1$ $\sqrt{\dfrac{F}{\alpha}} < W$

Case II $m^* \geq 1$ $\sqrt{\frac{F}{\alpha}} \geq W$

Case III $m^* < 1$

Case I) When $m^* \geq 1$ $\sqrt{\frac{F}{\alpha}} < W$

For small values of n, $m^*V^* < X/n$ and hence this will remain the optimal to the constrained problem This will continue as n is less than $n_1 = \frac{X\gamma}{\left(\sqrt{\frac{\beta^2\alpha}{4F}} - \alpha\right)}\sqrt{\frac{\alpha}{F}}$

At this value of n (n_1), $m^*V^* = X/n_1$ and for n larger than this value $m^*V^* > X/n$, on the boundary and (m^*, V^*) will be outside the feasible region

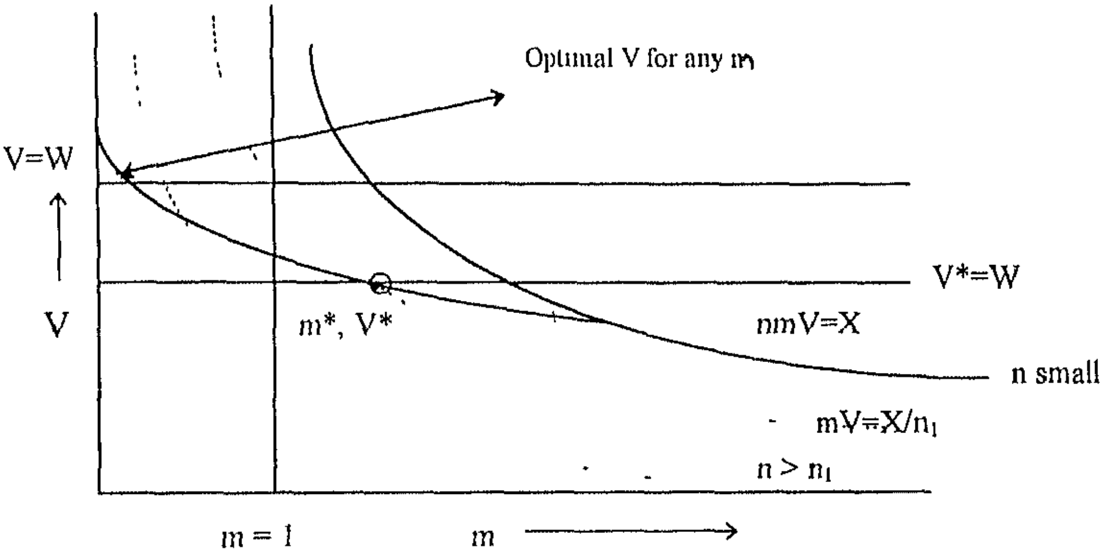


Figure 3.5. Optimal m and V when unconstrained optimal is the optimal

Beyond this point, we set $mV = X/n$ in Eqn 3 21 to get

$$m^* = \frac{X}{n}\sqrt{\frac{\alpha}{F}} \qquad (3\ 30)$$

$$V^* = \sqrt{\frac{F}{\alpha}} \qquad (3\ 31)$$

As m* and V* is outside the feasible region, the optimal solution is at one of the boundary i e point 4 or point 3 Further as there is only one maximum point in the region $m \geq 0, V \geq 0$, the value of optimal solution on the surface along $V_{opt} = \beta/(2(\alpha+\gamma m))$ will decrease in both the direction for the point 1 i e $Z_4 \leq Z_2 \leq Z_3$

Further $Z_3 \geq Z_2$ (as Z_3 is optimal on the boundary nmV=X) and hence $Z_3 \geq Z_4$ or Z_3 is the optimal point on the boundary

The present values of m^* and V^* are maintained till it reaches the boundary $m=1$ There the optimal shifts to $\{1, X/n\}$ This shift takes place at the second breakpoint which is achieved at $X\sqrt{\frac{\alpha}{F}}$ This condition is being shown in the fig 3 6 by case II

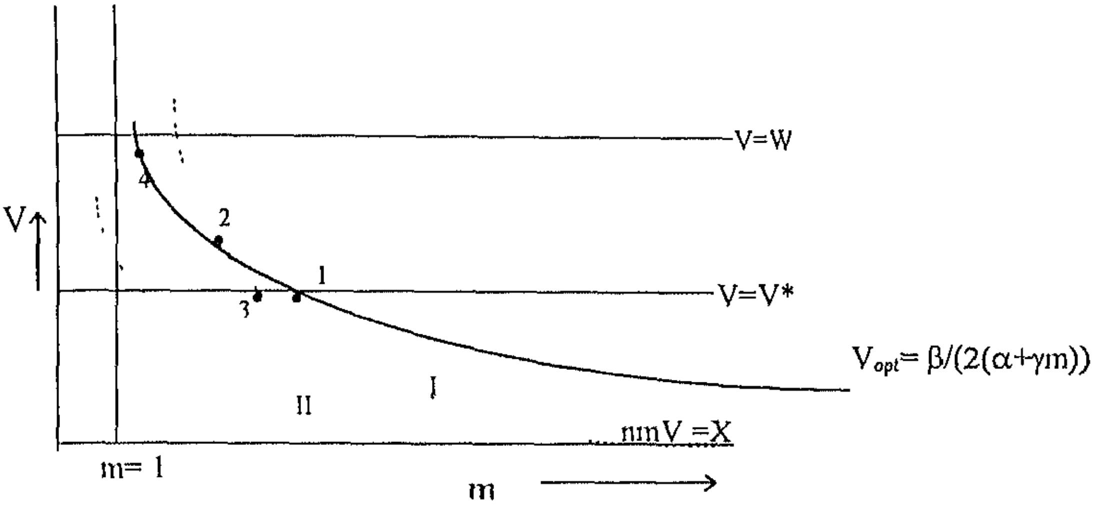


Figure 3.6 : Optimal m and V when Oversupply Constraint is Effective

Thus the variation of V_{opt} and m_{opt} with n for a given retailer is as given below fig 3 7

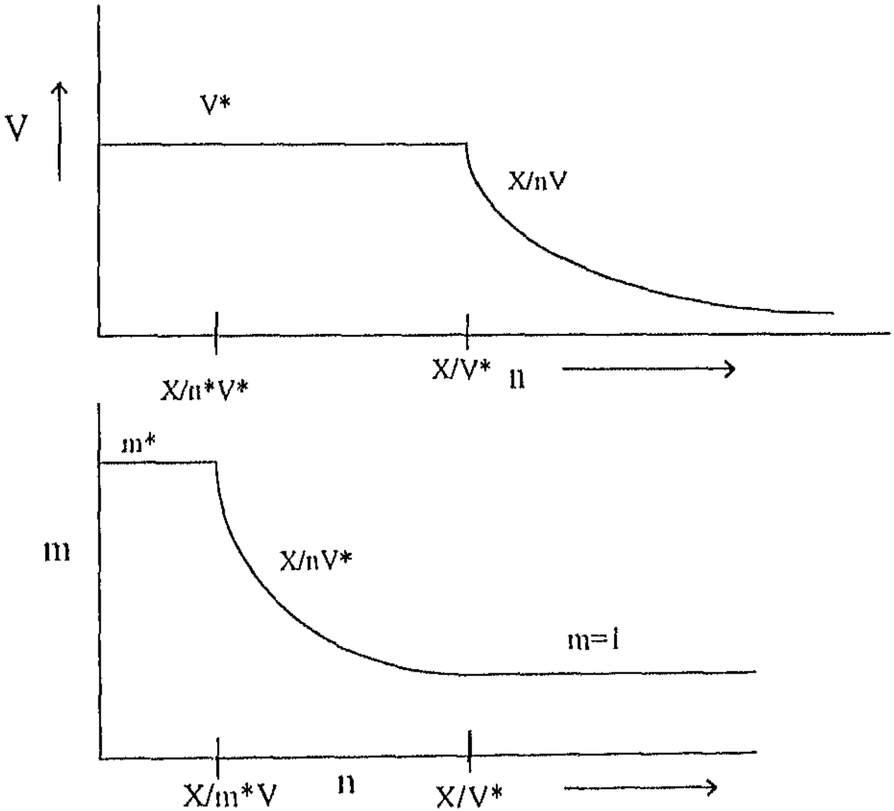


Figure 3.7: Variation of m_{opt} and V_{opt} with n

Case II) Now we consider the case when $W < \sqrt{\frac{F}{\alpha}}$ This case involves the vehicle capacity also

When n < $\frac{2\gamma WX}{\left(\beta W - \alpha W^2 - F\right)}$ then the oversupply constraint is always satisfied So we further subdivide this case about this point

i) When $n < \frac{2\gamma WX}{\left(\beta W - \alpha W^2 - F\right)}$

If we substitute V W in Eqn 3 23 we get the optimal value for this condition as

$$m_2^* = \frac{\beta W - \alpha W^2 - F}{2\gamma W^2} \tag{3 32}$$

This value of m_2^* always lies before the intersection of curves $V=W$ and $V=\frac{\beta}{2(\alpha+\gamma m)}$ as

$W < \sqrt{\frac{F}{\alpha}}$

The value of the objective function is optimal at point 1 in figure 3 6 without the constraint, so when the constraint is introduced we will try to move along the curve $V=\frac{\beta}{2(\alpha+\gamma m)}$ As the function is decreasing as we move further from point 1 we try to be as close as possible to point 1 on this curve i e at point 2, but as point 3 is always better than point 2 so 3 is the optimal point

so this value will give the optimal value

ii) When $n \geq \frac{2\gamma WX}{\left(\beta W - \alpha W^2 - F\right)}$ then $m^* = \frac{X}{nW}$. (3.33)

This solution remains optimal until the surface the reaches the boundary as defined by $m = 1$ This is achieved at $n = X/W$ Beyond this point $m^*=1$ and $V^*=X/n$

Thus the variation of V_{opt} and m_{opt} with n is as given below

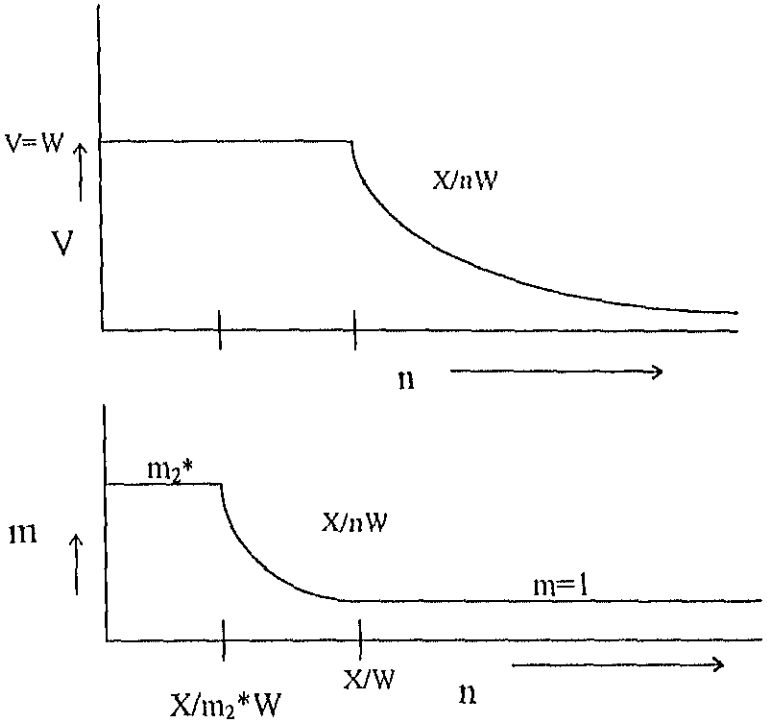


Figure 3.8 : Variation of m_{opt} and V_{opt} with n

Case III

Next we consider the case $m^* < 1$

Again we further subdivide this case also according to the vehicle capacity The vehicle capacity constraint becomes effective only when it cuts the line $m = 1$ below the point where m = 1 intersects with $V = \dfrac{\beta}{2(\alpha + \gamma m)}$ i e $\dfrac{\beta}{2(\alpha + \gamma)}$

When W is greater than this value then $m^* = 1$ and $V^* = \dfrac{\beta}{2(\alpha + \gamma)}$ but this value should also satisfy the oversupply constraint so if $n > \dfrac{2X(\alpha + \gamma)}{\beta}$ then $V^* = X/n$

When W is lesser than this value then $V^* = W$ and $m_2^* = \dfrac{\beta W - \alpha W^2 - F}{2\gamma W^2}$

As explained earlier this value of m_2^* lies before the intersection of the curve $V = \dfrac{\beta}{2(\alpha + \gamma m)}$ and $V = W$ but this m^* should also satisfy $m \geq 1$ constraint hence if $m_2^* \geq 1$ then $m_{opt} = m_2^*$ and the breakpoint is X/Wm_{opt} This value of (m^*, V^*) remains till the surface $nmV = X$ reaches $m = 1$ This is achieved at $n = X/W$ Beyond this point $m^* = 1$ and $V^* = X/n$ When $m_2^* \leq 1$ then the first breakpoint is missing while the second breakpoint remains the same

3.3 4 Solution Method for Problem P

In the section 3 3 2 we have concluded that for a given value of n for a retailer the optimal point will be one of the seven points combination We will refer to these seven points as type 1 to type 7 solutions The breakpoints are being referred to as B1 for the first breakpoint and B2 as the second one

We have further divided n into three ranges (1, B1), (B1, B2) and (B2, ∞) and have identified for each such range for a particular retailer the type of optimal solution (as a function of n) in that range

Objective Function

If in a given range of n, for a retailer the optimal solution is either of type 2, 3, 5 or 6 then the objective function is a linear function of n This implies that when all the retailers are such that they belong to this set then for maximising Z we will choose the highest possible value of n. When all the retailers' replenishment is of type two or four or seven then the contribution is of the nature $Z = A - nC - \frac{B}{n}$, where A and B are constants, and when the replenishment is of mixed nature i e some are being supplied at level independent of n and some at variable level then also the nature of contribution is again of type $Z = A - nC - \frac{B}{n}$, of course with different values of constants

Thus we have again $2k+1$ pairs of breakpoints as in the previous case Almost the same procedure is again employed to find the optimal decision variables These breakpoints are arranged in ascending manner and then pairs are made out of the adjacent numbers Now between these ranges we have optimal value of m and V and we have to find the optimal n These ranges will conform to one of the five cases discussed below

Case 1.

When all the values of m_j's and V_j's are both constant as in type (1,3,5,6) then, as explained earlier, to maximise the objective function value we will try to increase n as large as possible i e the upper limit of the range will be chosen as the optimal value of n

Case 2

When the optimal m_j's and V_j's are all following the relationship

$m_j^* = \frac{X_j}{n}\sqrt{\frac{\alpha_j}{F_j}}$ and $V_j^* = \sqrt{\frac{F_j}{\alpha_j}}$ for a given range, then on substituting these values in the original objective function we can get it as a function of n

$$Z(n) = \left[(R)\sum_{j=1}^{k} X_j - \frac{hp}{2P}\sum_{j=1}^{k}(1-\phi_j)X_j\sqrt{\frac{F_j}{\alpha_j}} - \frac{1}{2}\sum_{j=1}^{k}(1-\phi_j)\frac{hr_j X_j}{P}\sqrt{\frac{F_j}{\alpha_j}} - \frac{hp}{2}\sum_{j=1}^{k}\phi_j\sqrt{\frac{F_j}{\alpha_j}}\right] -$$

$$\left[\frac{1}{2}\sum_{j=1}^{k}\phi_j hr_j\sqrt{\frac{F_j}{\alpha_j}} + \sum_{j=1}^{k}X_j\sqrt{\alpha_j F_j} + \sum_{j=1}^{k}X_j f_j + ht\sum_{j=1}^{k}X_j T_j\right] - nC -$$

$$\frac{1}{n}\left[\frac{1}{2}\left(\sum_{j=1}^{k}(1-\phi_j)hr_j X_j\frac{(P-X_j)}{P} + hp\sum_{j=1}^{k}\phi_j X_j\frac{(P-X_j)}{P}\right)\right] \qquad (3.35)$$

Writing the above equation in a simplified form we get

$$Z(n) = A - nC - \frac{B}{n} \qquad (3.36)$$

$A =$

$$\left[(R)\sum_{j=1}^{k} X_j - \frac{hp}{2P}\sum_{j=1}^{k}(1-\phi_j)X_j\sqrt{\frac{F_j}{\alpha_j}} - \frac{1}{2}\sum_{j=1}^{k}(1-\phi_j)\frac{hr_j X_j}{P}\sqrt{\frac{F_j}{\alpha_j}} - \frac{hp}{2}\sum_{j=1}^{k}\phi_j\sqrt{\frac{F_j}{\alpha_j}}\right] -$$

$$\left[\frac{1}{2}\sum_{j=1}^{k}\phi_j hr_j W_j + \sum_{j=1}^{k}\frac{F_j X_j}{W_j} + \sum_{j=1}^{k}X_j f_j\right]$$

$$B = \left[\frac{1}{2}\left(\sum_{j=1}^{k}(1-\phi_j)hr_j X_j\frac{(P-X_j)}{P} + hp\sum_{j=1}^{k}\phi_j X_j\frac{(P-X_j)}{P}\right)\right]$$

$$\frac{\partial Z}{\partial n} = -C + \frac{B}{n^2} \qquad (3.37)$$

$$\frac{\partial^2 Z}{\partial n^2} = -\frac{B}{n^3} \quad (<0) \text{ hence the first derivative gives the maxima} \qquad (3.38)$$

$$\Rightarrow n^* = \sqrt{\frac{B}{C}} \qquad (3.39)$$

Case 3

When all the retailers have their optimal m and V according to the relationship $m=\frac{X}{nW}$ and $V=W$, then substituting these values in the original objective function we get

$$Z(n)=\left[(R)\sum_{j=1}^{k}X_j-\frac{hp}{2P}\sum_{j=1}^{k}(1-\phi_j)X_jW_j-\frac{1}{2}\sum_{j=1}^{k}(1-\phi_j)\frac{h_jX_j}{P}W_j-\frac{hp}{2}\sum_{j=1}^{k}\phi_jW_j\right]-$$

$$\left[\frac{1}{2}\sum_{j=1}^{k}\phi_jh_jW_j+\sum_{j=1}^{k}\frac{F_jX_j}{W_j}+\sum_{j=1}^{k}X_jf_j\right]-nC-$$

$$\frac{1}{n}\left[\frac{1}{2}\left(\sum_{j=1}^{k}(1-\phi_j)h_jX_j\frac{(P-X_j)}{P}+hp\sum_{j=1}^{k}\phi_jX_j\frac{(P-X_j)}{P}\right)\right] \qquad (3\ 40)$$

This is almost the same as in above case and by similar analysis we get $n^*=\sqrt{\frac{B}{C}}$ where

$$B=\left[\frac{1}{2}\left(\sum_{j=1}^{k}(1-\phi_j)h_jX_j\frac{(P-X_j)}{P}+hp\sum_{j=1}^{k}\phi_jX_j\frac{(P-X_j)}{P}\right)\right]$$

Now we check the range for which this case is true

If $n^* >$ upper limit of the range then n_{opt} = upper limit

If $n^* <$ lower limit of the range then n_{opt} = lower limit else $n_{opt}=n^*$.

Case 4

When all the retailers have their optimal m and V according to the relationship $m=1$ and $V=X/n$ then substituting these values in the original objective function we get again a function of the nature $Z(n)=A-nC-\frac{B}{n}$ and the n* is similarly obtained as in the above case

Case 5

When the replenishment is of mixed nature, i e some are at constant level and some are at full replenishment level, is being considered in this case

Let the contribution from the constant replenishments be $nD+B^*$, then following the above analysis the objective function can be expressed as

$$Z=A+n(D-C)-\frac{B}{n} \qquad (3\ 41)$$

When $D-C>0$ then the objective function value increases with increase in n so we choose the upper limit of n in this case

When $(D-C)<0$ then again following the above analysis we can easily obtain

$$n^* = \sqrt{\frac{B}{C-D}} \tag{3.42}$$

Now we check the range for which this case is true

If n^* > upper limit of the range then n_{opt} = upper limit

If n^*< lower limit of the range then n_{opt} = lower limit else $n_{opt}=n^*$

3.4 Algorithm

First, the retailerwise subproblem is solved independently and then the original problem is dealt with

I Subproblem

a) Calculate α_j, β_j and γ_j

$$\alpha_j = (1-\phi_j)\left(\frac{hp}{2P}+\frac{hr_j}{2P}\right)+\phi_j\left(\frac{hp+hr_j}{2X_j}\right),\quad \beta_j = R+s-f_j-ht\,T_j, \text{and}$$

$$\gamma_j = (1-\phi_j)\left(\frac{P-X_j}{2PX_j}\right)hr_j+\phi_j\left(\frac{P-X_j}{2PX_j}\right)hp$$

b) Calculate $V_j^* = \sqrt{\frac{F_j}{\alpha_j}}$, $m_j^* = \frac{\sqrt{\frac{\beta^2\alpha}{4F}}-\alpha}{\gamma}$

c) Calculate the breakpoints with the help of the decision table 3.1 given below

II Problem

a) Combine the arrays of breakpoints and then arrange the series in an ascending manner

b) Make pairs out of the above arranged list from each consecutive numbers to construct ranges

c) Check the type of each range

d) Calculate the n_{opt} and the corresponding m_{opt}, V_{opt} and objective function value for each range

e) The range giving the best objective function value is chosen as the optimal range and the decision variables for this range are the optimal decision variables

When $m_u^* \geq 1$

Condition	B1	B2	0 - B1	B1 - B2	B2 -∞	
$V_u^* < W$	$X/m_u^*V_u^*$	X/V_u^*	m_u^*	X/nV_u^*	1	m_{opt}
$m_u^* \geq 1$			V_u^*	V_u^*	X/n	V_{opt}
$V_u^* \geq W$	X/m_w^*W	X/W	m_w^*	X/nW	1	m_{opt}
$m_u^* \geq 1$			W	W	X/n	V_{opt}

When $m_u^* < 1$ then

Condition	B1	0 - B1	B1 - ∞	
$V^* < W$	X/V_V^*	1	V_V^*	m_{opt}
		1	X/n	v_{opt}
$V^* > W$	X/W	$\max(m_w, 1)$	W	m_{opt}
		1	X/n	v_{opt}

Table 3.1 : Decision Table for m_{opt} and V_{opt}

where

$$m_w = \frac{\beta W - \alpha W^2 - F}{2\gamma W^2}$$

$$V_v^* = \sqrt{\frac{F}{\alpha}}$$

$$V_v^* = \frac{\beta}{2(\alpha + \gamma)}$$

$$m_u^* = \frac{\sqrt{\frac{\beta^2 \alpha}{4F}} - \alpha}{\gamma}$$

3.5 Case When m is Required to be Integer

Unconstrained Case

In the case m* and V* is within the feasible region Further as the function is concave on the surface with respect to m

$V_{opt} = \beta/(2(\alpha+\gamma m))$

Hence the optimal solution will be $\lfloor m^* \rfloor$ or $\lceil m^* \rceil$ and $V_{opt} = \beta/(2(\alpha+\gamma\lfloor m^* \rfloor)$

or $V_{opt} = \beta/(2(\alpha+\gamma\lceil m^* \rceil)$

In case $\lceil m^* \rceil$ will fall outside the feasible region, optimal point will be on the boundary

Constrained Case

As the function is concave along the boundary, the optimal integer solution will be rounded up or down value of m^* and appropriately adjusted V_{opt}

3.6 Complexity Analysis

Let k indicate the number of retailers In the first part, **SP**, each of the five step requires a constant number of operations (addition, subtraction, comparison, division etc) so the total number of operations for one retailer is a constant, say L This procedure is repeated k number of times thus the number of operations required is Lk assuming that the square root operation takes a constant amount of time i e operation time is independent of problem size

For the problem **P** we have to first sort the $2k$ ranges which requires ($2k \log 2k$) steps Now for each range we make k evaluations to find the optimal decision variables This step requires $2k^2$ steps

Thus the problem requires $Lk+2k \log 2k+ 2k^2$ steps Hence the complexity of the above algorithm is $O(k^2)$i e it is a polynomial function of number of retailers

3.7 Flow Chart

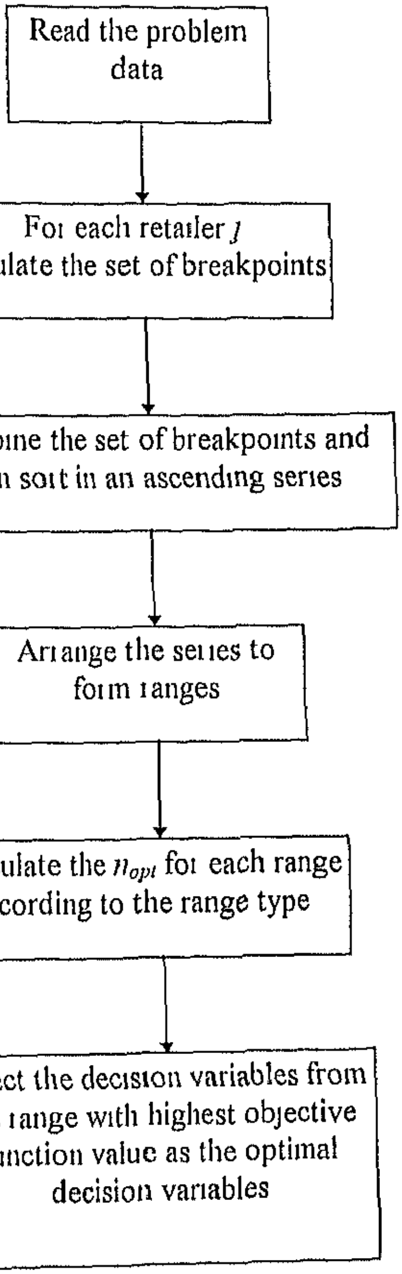

Read the problem data
For each retailer j calculate the set of breakpoints
Combine the set of breakpoints and then sort in an ascending series
Arrange the series to form ranges
Calculate the n_{opt} for each range according to the range type
Select the decision variables from the range with highest objective function value as the optimal decision variables

Figure 3.9 : Flow Chart for the Algorithm

3.8 Simultaneous Production-Multiple Replenishment to each Retailer

We consider a situation where production is taking place throughout the production cycle and shipments to a retailer are made at equally spaced intervals throughout the cycle. It may be shipment to minimise the retailer inventory holding cost Now we will formulate the net profit function for this case

3.8.1 Problem Formulation

A) Production related cost

The following figures represents inventory status diagram at the production site The relevant costs are cost of production setup and inventory costs

1) Production set up cost There are n production setups in a unit time As the cost of set up is C, the total production set up cost per unit time is nC (3 43)

2) Inventory holding cost The total inventory holding cost in one period can be computed by measuring the area of the above figure If the inventory holding cost is hp then the total inventory cost per unit period for this is $n\ hp\ \frac{1}{2}\sum_{j=1}^{k}\frac{m_j V_j^2}{P}$

$$= \frac{n}{2}\ \frac{hp}{P}\sum_{j=1}^{k} m_j V_j^2 \qquad (3\ 44)$$

B) Transportation related cost

1) Cost of transportation

A fixed freight charge F_j and a variable cost f_j per unit transported is incurred depending upon the retailer Thus the total transportation cost is

$$n[\sum_{j=1}^{k} F_j m_j + \sum_{j=1}^{k} m_j f_j V_j\ ] \qquad (3\ 45)$$

2) Transportation inventory holding cost

If the transportation time is T_j for the j^{th} retailer and the transportation inventory holding cost is ht then the total inventory holding cost during transportation over the unit time period is $n\ ht\sum_{j=1}^{k} m_j V_j T_j$ (3 46)

C) Revenue and cost at the retailer site

1) Revenue

The total amount of demand satisfied in one period is $n\sum_{j=1}^{k} m_j V_j$ The total revenue generated is $n\,R\,\sum_{j=1}^{k} m_j V_j$ (3 47)

2) Inventory holding cost

The following figures represents the stock position at any time at the retailer outlet

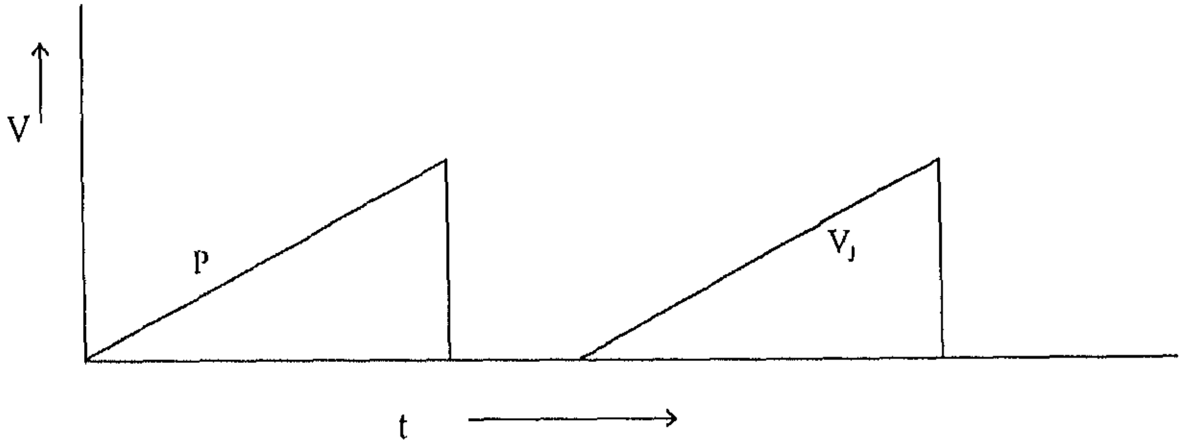


Figure 3.10 : Inventory Status Diagram at Production Site

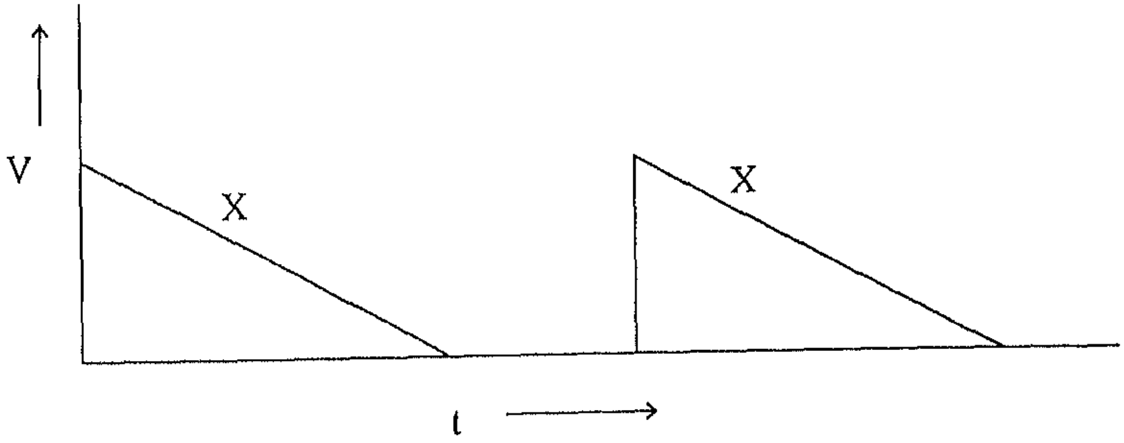


Figure 3.11 Inventory Status Diagram at Retailer Site

If the per unit holding cost at the retailer's site is hr_j, the total inventory holding cost per unit time period is

$$n\,\frac{1}{2}\sum_{j=1}^{k}\frac{hr_j m_j V_j^2}{X_j} \qquad (3\ 48)$$

3) Shortage cost

The number of lost sales in one period at a retailer j is $(X_j - n\ m_j V_j)$ If the per unit shortage cost is s then the shortage cost per unit time period is

$$s\sum_{j=1}^{k}(X_j - nm_jV_j) \qquad (3\ 49)$$

The profit function is obtained by subtracting all costs from the revenue obtained The profit is a function of n, m_j and V_j's Let us represent this function by Z_l Then this problem **OP** can be stated as following

Problem OP

$$Z(n, m_j, V_j) = n\,R\sum_{j=1}^{k} m_jV_j - nC - \frac{n}{2}\,\frac{hp}{P}\sum_{j=1}^{k} m_jV_j^2 - n[\sum_{j=1}^{k} F_jm_j + \sum_{j=1}^{k} m_jf_jV_j] - n\,ht\sum_{j=1}^{k} m_jV_jT_j$$

$$-\,n\,\frac{1}{2}\sum_{j=1}^{k}\frac{hr_jm_jV_j^2}{X_j} - s\sum_{j=1}^{k}(X_j - nm_jV_j) \qquad (3\ 50)$$

We will maximise Z to obtain the optimal policy The feasible policy has to further satisfy the following constraints

1) Supply is less than or equal to demand

$$n\ m_j V_j \le X_j \qquad (3\ 51)$$

2) The volume shipped is less than or equal to the vehicle capacity

$$V_j \le W_j \qquad (3\ 52)$$

3) Only one vehicle is allotted per station so the production cycle must be so spaced that the vehicle can return to take the new replenishment

$$n\ m_j \le \min[\frac{1}{T_j}] \qquad (3\ 53)$$

4) non negativity constraint

$$n,\ m_j V_j \ge 0 \qquad .(3\ 54)$$

3.8.2 Solution Methodology

The eqn 3 50 can be written as a function of n as

$$Z(n, m_j, V_j) = n\left[\sum_{1}^{k}(R - f_j - htT_j + s)m_jV_j - \sum_{j=1}^{k}\left(\frac{hp}{2P} + \frac{hn_j}{2X_j}\right)m_jV_j^2 - \sum_{j=1}^{k} m_jF_j - C\right] - s\sum_{j=1}^{k} X_j$$

$$= n\left[\sum_{j=1}^{k}(\beta_j m_jV_j - \alpha_j m_jV_j^2 - F_jm_j) - C\right] - s\sum_{j=1}^{k} X_j \qquad (3\ 55)$$

where $\beta_j = R - f_j - htT_j + s$ *and* $\alpha_j = \dfrac{hp}{2P} + \dfrac{hn_j}{2X_j}$

subject to the constraints as given by Eqn 3 51 to 3 54

It can be observed that for a selected value of n the objective function can be written as $Z(V) = \sum_{j=1}^{k} V_j$ Further there are no constraint binding the different decision variables together so the problem can be decomposed in k independent subproblems, each of the given type.

Subproblem OSP

The reduced j^{th} subproblem is

$$\max Z_1(m, V) = m(\beta V - \alpha V^2 - F) \qquad (3\ 56)$$

This function indicates that for the constraints given above, the function will try to be on the surface nmV=X Substituting this value of m in eqn 3 56 we obtain

$$Z_1 = \frac{1}{n}\left(X\beta - \alpha V - \frac{FX}{V}\right) \qquad (3\ 57)$$

$$\frac{\partial Z_1}{\partial V} = -\alpha + \frac{FX}{V^2} \qquad (3\ 58)$$

$$\frac{\partial^2 Z_1}{\partial V^2} = -\frac{2FX}{V^3} \qquad (\text{<0 hence the function is concave}) \qquad (3\ 59)$$

$$V^* = \sqrt{\frac{FX}{\alpha}} \qquad \text{and } m^* = \sqrt{\frac{X\alpha}{F}} \qquad (3\ 60)$$

CENT BRARY

This value of m^* and V^* will remain optimal till this curve ($nmV = X$) touches the boundary at $m = 1$ This shift takes place at the value $n = X/V^*$ from where $m^*=1$ and $V^* = X/n$

If the value of V^* obtained in Eqn 3 60 is greater than W then $V^*=W$ and $m^*=X/nW$ and similarly as the above case this value would remain till it reaches m = 1 boundary This intersection takes place at $n = X/W$ from where $m^*=1$ and $V^*=X/n$

Now the value obtained for m^* in Eqn 3 60 can be below 1 which is not permissible For this case we would try to remain on the curve $nmV=X$ and reach upto $m = 1$ on this curve So the optimal will be $m^*= 1$ and $V^* = X/n$ and if this value is also greater than W then we set $V^*=W$

Problem OP

Thus from the above analysis we can get the m^* and V^* at any value of n for each retailer The set of possible m^* and V^* are

1	$m^*=\sqrt{\dfrac{X\alpha}{F}}$	$V^*=\sqrt{\dfrac{FX}{\alpha}}$
2	$m^*= X/nW$	$V^*=W$
3	$m^*=1$	$V^*=X/n$
4	$m^*=1$	$V^*= W$

This value shifts to other kind at a breakpoint The breakpoints have also been discussed above First the ranges are determined and then the breakpoints are arranged in an ascending manner to obtain continuos function between the breakpoints The solution methodology then follows the earlier two cases First the range type is determined and then the local optimal is calculated for that range From the permissible set of m^* and V^* it is clear that the overall objective function will be of the kind $A-nC-B/n$ The rest of solution methodology is exactly the same

3.9 Example

To illustrate the above analysis we consider a numerical example with the following data

C= \$ 1 000 000 per production set up, R = \$200 per unit, P = 30 000 units per year, s = \$600 per unit, hp = \$100 per unit per year, ht = \$100 per unit per year

Data assumed to be common for all retailers

(hr = \$200 per unit per year, W = 2000 units, f= \$2 per unit, F = \$100 000 per transportation)

$X_1 = 12\ 000$ units $T_1 = 0.08$ year

$X_2 = 6\ 000$ units $T_2 = 0.05$ year

$X_3 = 9\ 000$ units $T_3 = 0.10$ year

Solution

Initially the retailerwise subproblem is solved and the following values are obtained

	Retailer 1	Retailer 2	Retailer 3
α	0.0125	0.0125	0.0167
β	790	793	788
γ	0.002	0.0067	0.00369
B1	0.0839	0.115	0.099
B2	6.00	3.00	4.50
m*	63.57	19.05	37.0
V*	2828	2828	2447

Table 3.2 Subproblem Calculation

The breakpoints are then arranged in an ascending manner to get piecewise continuous function Between these breakpoints, the optimal value of *n* is determined and the corresponding m_{opt} and V_{opt} are calculated with the help of the decision table

The value of the objective function is calculated for each range A summary of the results for the integrated problem is given below in Table 3.3

Range	Optimal n	Objective function
0 - 0 0839	0 0839	-7806154
0 0839 -0 099	0 099	-6392449
0 099 -0 115	0 115	-5161844
0 115 - 3 00	0 956	456276 8
3 00 - 4 50	3 00	-499999
4 50 - 6 00	4 50	-3983332

Table 3.3: Problem Calculation

As clearly evident from the above table, the highest objective function is obtained at n=0 956

The corresponding decision variables are

$m_1=6$ $V_1 = 2000$

$m_2 = 3$ $V_2 = 2000$

$m_3 = 5$ $V_3 = 1881$

Chapter 4

CONCLUSION

4.1 Conclusion

The present work dealt with the problem of obtaining optimum transportation policy for a production distribution system with one production centre and multiple retailers by analysing the trade-offs between production, inventory and transportation costs

Initially the situation when only one replenishment is being made to a retailer in a single production cycle was considered The problem was solved by decomposing it to retailerwise subproblem to find the optimal replenishment volume for a retailer for a given number of production set-ups The results indicated that the objective function was a piecewise continuous function of number of set-ups with the discontinuities at the points where replenishment policy changes for a retailer Between all the breakpoints the optimal number of production set-ups and the corresponding objective function value is calculated and the range which gives the best objective function value gives the optimal decision variables

Later the case with multiple replenishment was analysed For this situation two types of policies single set for each retailer and multiple set-up for each retailer were analysed In both the cases, the problem was first formulated and then solved on the lines of the single replenishment case First it was decomposed into retailerwise subproblem and then solved independently Finally the subproblems were integrated to get the overall objective function which was then solved by dividing it into continuos ranges as in the previous case

The algorithms presented to solve the problems have the complexity of $O(k^2)$, where k is the number of the retailers i e, it is a polynomial function of the number of retailers under the assumption that square root operation of a number can be determined in constant time

4.2 Scope for future work

The present work can be developed in the following areas

1 The case when only when product is being supplied with only one production centre has been considered This model can be generalised by formulating the problem for multiple origins with multiple products

2 The demand was assumed to be deterministic Stochastic demands may be an interesting area for further research

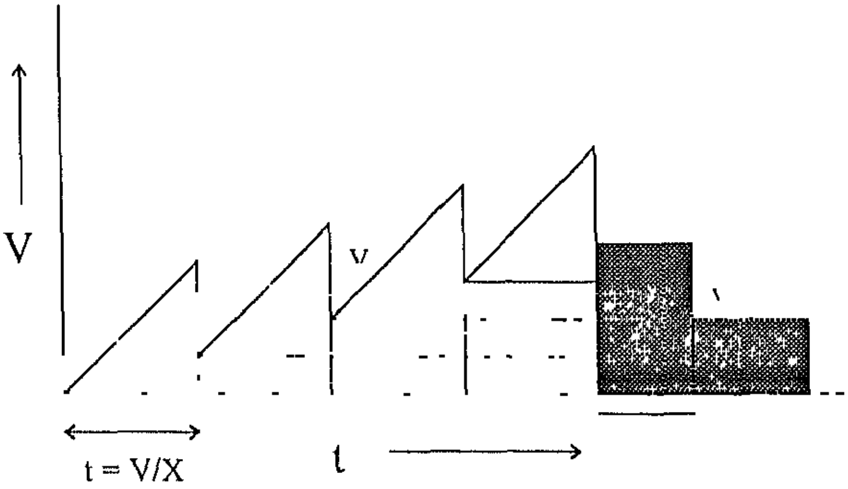
3 The problem can also be analysed for the condition where the production rate is lower than the total demand rate

4 Integration of vehicle routing component will also pose a challenging work

Appendix

A.1.

Area of the diagram representing the inventory status at production site



Inventory Status Diagram at Production Site for Case II

The area can be divided into three parts the triangles, the rectangles starting from the point where the production has stopped and the third part is of the rectangles that fall below the triangles

Area of first part

Number of triangles = $\frac{mX}{P}$ Let the number be denoted by b

Area of triangles = $\frac{1}{2}\frac{PV^2b}{X^2}$

Area of the second part

Area of a slab = $\left(\frac{PV}{X}-V\right)\frac{V}{X}$

Number of slabs = $b(b-1)/2$

Hence the area = $\frac{V^2}{2X^2}(P-X)b(b-1)$

Area of the third part

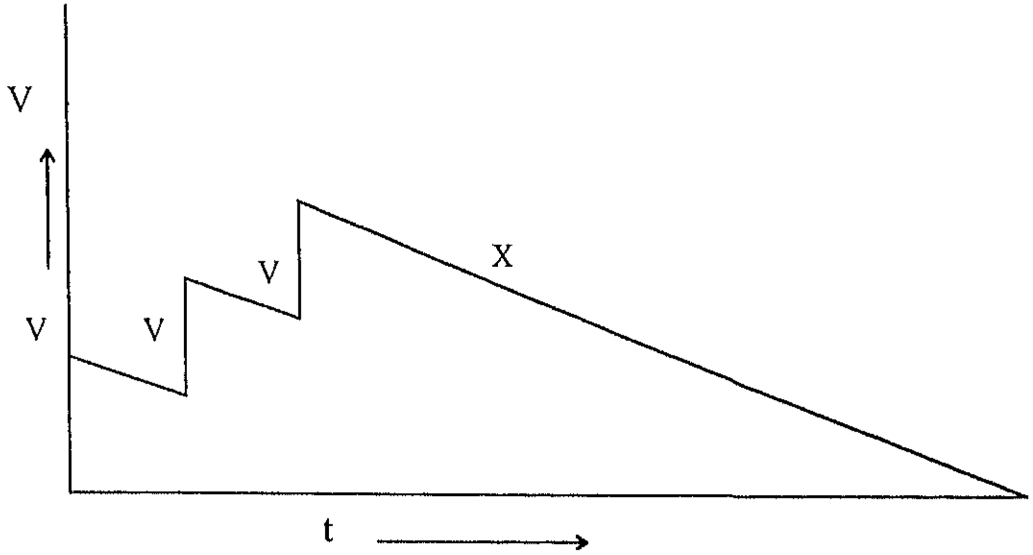
The number of blocks after the production has stopped is $\frac{m(m+1)}{2} - b$

Hence the area of this part is $= \left[\frac{m(m+1)}{2} - b\right]\frac{V^2}{X}$

The total area is calculated by adding all the three parts obtained above

The total area is thus $\frac{mV^2}{X}\left[(m+1) - \left(\frac{mX}{P}\right)\right]$

A.2



Area of the total figure $= \frac{1}{2}\frac{m^2V^2}{X}$

Area of the part bounded by dotted lines $= \frac{m(m-1)V^2}{2P}$

Thus the area of the region of interest is $mV^2\left(\frac{m}{X}-\frac{m-1}{P}\right)$

References

1 Anand, V (1997), "Optimal Transport Policy for Production-Distribution System", M Tech Dissertation, Industrial and Management Engineering Department, Indian Institute of Technology Kanpur

2 Anily, S and A Federgruen (1990), "One Warehouse Multiple Retailer Systems with Vehicle Routing Costs", *Management Science*, Vol 36, No 1, 92-114

3 Anily, S and A Federgruen (1993), "Two-Echelon Distribution Systems with Vehicle Routing Costs and Central Inventories", *Operations Research*, Vol 41, 37-47

4 Benjamin, J (1989), "An Analysis of Inventory and Transportation Costs in a Constrained Network", *Transportation Science*, Vol 23,No 3, 177-183

5 Burns, L D , R W Hall and D E Blumenfield (1985), "Distribution Strategies that Minimise Transportation and Inventory Costs", *Operations Research*, Vol 33, No 3, 469-488

6 Chein, T W , A Balakrishnan and R T Wong (1989), "An Integrated Inventory Allocation and Vehicle Routing Problem", *Transportation Science*, Vol 23, 67-76

7 Daganzo, C F , D E Blumenfield, L D Burns and J D Diltz (1985), "Analysing Trade-offs between Transportation Inventory and Production Costs on Freight Networks", *Transportation Research*, 19B, 361-380

8 Federgruen, A and P Zipkin (1984), "A Combined Vehicle Routing and Inventory Allocation Problem", *Operations Research*, Vol 32, No 5, 1019-1037

9 Gallego, G and Simchi-Levi (1990), "On the Effectiveness of Direct Shipping Strategy for One-Warehouse Multi-Retailer *R*-System", *Management Science*, Vol 36, No 2, 240-243

10 Hall, R W (1992), "A Note on Bound for Direct Shipping Cost", *Management Science*, Vol 38, No 8, 1212-1214

11 Lee, H and C Billington (1993), "Materials Management in Decentralised Supply Chains", *Operations Research*, Vol 41, No 5, 853-847.

12 Vishwanathan, S and K Mathur (1997),"Integrating Routing and Inventory Decisions in One Warehouse Multi-Retailer Multi-Product Distribution System", *Management Science,* Vol 43, No 3, 294-312

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